Efficient computation of expected allocations

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December 14 2022, AFI Seminars

Motivation

- Insurance: risk pooling mechanism
- Insurance companies
  - Compute risk measures at the portfolio level
  - Must compute the contribution of each risk to the portfolio (risk measures)
- Peer-to-peer insurance
  - Pools risks together
  - Must compute the contribution of each participant to the pool (risk-sharing rules)
- Possible difficulties:
  1. Large pools / portfolios
  2. Heterogeneous risks
  3. Dependent risks

Motivation

Peer-to-peer insurance

- Peer-to-peer insurance pricing schemes: compute the contribution of participant according to risk sharing rule [Denuit, 2020a]
- Conditional mean risk sharing rule [Denuit and Dhaene, 2012]: popular choice
- Satisfies desirable properties [Denuit et al., 2022]
- Axiomatic justification [Jiao et al., 2022]
- Price for the ith participant is expected contribution of risk X_i given that the actual loss S was $k > 0$

$$E[X_i|S = k] = \frac{E\left[X_i \times 1_{\{S=k\}}\right]}{\Pr(S = k)}, \quad i \in \{1, \ldots, n\}$$

- Consider a portfolio of risks with overall risk measure $TVaR_\kappa(S)$, $\kappa \in (0,1)$, where $E[S] < \infty$
- Problem: determine allocation (risk contribution) of risk X_i, $i \in \{1,\ldots,n\}$, to overall risk measure $TVaR_\kappa(S)$
- TVaR-based capital allocation with Euler rule:

$$TVaR_\kappa(X_i;S) = \frac{E\left[X_i \times 1_{\{S>k_0\}}\right] + E\left[X_i \times 1_{\{S=k_0\}}\right]\beta}{1-\kappa}, \tag{1}$$

where $k_0 = VaR_\kappa(S)$,

$$\beta = \begin{cases} \frac{\Pr(S\leq k_0)-\kappa}{\Pr(S=k_0)}, & \text{if } \Pr(S=k_0) > 0 \\ 0, & \text{otherwise} \end{cases},$$

$$E\left[X_i \times 1_{\{S>k_0\}}\right] = E[X_i] - E\left[X_i \times 1_{\{S\leq k_0\}}\right]$$

More generally, RVaR-based capital allocation with Euler rule: for for $\alpha_1 < \alpha_2$,

$$\mathrm{RVaR}_{\alpha_1,\alpha_2}(X_i;S) = \frac{1}{\alpha_2-\alpha_1}\left(E\left[X_1 \times 1_{\left\{S=F_S^{-1}(\alpha_1)\right\}}\right]\frac{F_S\left(F_S^{-1}(\alpha_1)\right)-\alpha_1}{\Pr\left(S=F_S^{-1}(\alpha_1)\right)}\right.$$
$$+E\left[X_1 \times 1_{\left\{F_S^{-1}(\alpha_1)<S\le F_S^{-1}(\alpha_2)\right\}}\right]$$
$$\left.+E\left[X_1 \times 1_{\left\{S=F_S^{-1}(\alpha_2)\right\}}\right]\frac{\alpha_2-F_S\left(F_S^{-1}(\alpha_2)\right)}{\Pr\left(S=F_S^{-1}(\alpha_2)\right)}\right)$$

Motivation

- Consider a portfolio of n risks $\boldsymbol{X} = (X_1, \dots, X_n)$
- Support for each rv: $h\mathbb{N}_0 = \{0, h, 2h, \dots\}$, with $h > 0$
- Let S be the aggregate loss rv, that is, $S = \sum_{j=1}^{n} X_i$

Objective of talk

1. Provide convenient representations for the values of $E\left[X_i \times 1_{\{S=kh\}}\right]$ and $E\left[X_i \times 1_{\{S\leq kh\}}\right]$ for $i \in \{1, \dots, n\}$ and $k \in \mathbb{N}_0$
2. Provide efficient computation methods for $E\left[X_i \times 1_{\{S=kh\}}\right]$ and $E\left[X_i \times 1_{\{S\leq kh\}}\right]$

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Common approaches

Common approaches

- For the remainder of this talk, set $h = 1$
- Direct computation. Let $S_{-i} = \sum_{j=1, j\neq i}^{n} X_j$, then
  - Discrete case:

$$E\left[X_i \times 1_{\{S=k\}}\right] = \sum_{j=0}^{k} j f_{X_i, S_{-i}}(j, k-j)$$

$$E\left[X_i \times 1_{\{S\leq k\}}\right] = \sum_{j=0}^{k} E\left[X_i \times 1_{\{S=j\}}\right]$$

  - Continuous case

$$E\left[X_i \times 1_{\{S=x\}}\right] = \int_0^x z f_{X_i, S_{-i}}(z, x-z)\, \mathrm{d}z$$

$$E\left[X_i \times 1_{\{S\leq x\}}\right] = \int_0^x E\left[X_i \times 1_{\{S=y\}}\right] \mathrm{d}y$$

- Used in, for instance, in [Bargès et al., 2009]

Common approaches

- Size-biased transform: expected allocation is the expected value of a size-bias transformed rv
- Size-biased transform in actuarial science: [Denuit, 2019], [Denuit, 2020b]
- Expected allocations and cumulative expected allocations: [Landsman and Valdez, 2003], [Furman and Landsman, 2005], [Furman and Landsman, 2008], [Denuit and Robert, 2020], [Denuit and Robert, 2021c], [Denuit and Robert, 2021b], [Denuit and Robert, 2021a]

Ordinary generating functions

Generating functions

- Method relies on generating functions

Definition (Ordinary generating function)

For a sequence $\{a_k\}_{k\geq 0}$, the function

$$A(z) = \sum_{k=0}^{\infty} a_k z^k$$

is its ordinary generating function (OGF). We use the notation $[z^k]A(z)$ to refer to the coefficient a_k, $k \in \mathbb{N}_0$.

- Generating functions are a "bag" which hold the values of a sequence in a single formula [Sedgewick and Flajolet, 2013]

Probability generating function

- Important generating function for actuaries: probability generating function
- Probability generating functions are OGFs for the sequence of probability masses, that is,

$$\mathcal{P}_X(t) = E\left[t^X\right] = \sum_{k=0}^{\infty} t^k \Pr(X = k)$$

- One can recover the values of $\Pr(X = k)$ by differentiating

$$[t^k]\mathcal{P}_X(t) = \Pr(X = k) = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathcal{P}_X(t)\bigg|_{t=0}, \quad k \in \mathbb{N}_0$$

- Alternatively, use fast Fourier transform (FFT) algorithm

Probability generating function

- Multivariate probability generating function

$$\mathcal{P}_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = E\left[t_1^{X_1} \times \cdots \times t_n^{X_n}\right]$$

- When risks are independent,

$$\mathcal{P}_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \prod_{j=1}^{n} \mathcal{P}_{X_j}(t_j)$$

- For aggregate risk

$$\mathcal{P}_S(t) = E\left[t^S\right] = E\left[t^{X_1+\cdots+X_n}\right] = \mathcal{P}_{X_1,\ldots,X_n}(t,\ldots,t)$$

Properties of ordinary generating functions

The following properties of generating functions hold (see [Sedgewick and Flajolet, 2013])

1. Addition $A(z) + B(z) = \sum_{k=1}^{\infty}(a_k + b_k)z^k$.
2. Right shift $zA(z) = \sum_{k=1}^{\infty} a_{k-1}z^k$.
3. Index multiply $A'(z) = \sum_{k=0}^{\infty}(k+1)a_{k+1}z^k$.
4. Convolution $A(z)B(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) z^k$.
5. Partial sum $A(z)/(1-z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j\right) z^k$.

Main results

Main result 1

- We present a generating function for the allocations of risk X_1

Theorem

We have

$$\mathcal{P}_S^{[1]}(t) := \left[t_1 \times \frac{\partial}{\partial t_1} P_{\mathbf{X}}(t_1, t_2, \ldots, t_n) \right]_{t_1 = \cdots = t_n = t} = \sum_{k=0}^{\infty} t^k E\left[X_1 \times 1_{\{S=k\}} \right].$$

Then, $\mathcal{P}_S^{[1]}(t)$ is the OGF for the sequence of expected allocations $\{E\left[X_1 \times 1_{\{S=k\}}\right]\}_{k \in \mathbb{N}_0}$.

- One can recover the values of $E\left[X_1 \times 1_{\{S=k\}}\right], k \in \mathbb{N}_0$ by differentiating

$$[t^k]\mathcal{P}_S^{[1]}(t) = E\left[X_1 \times 1_{\{S=k\}}\right] = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathcal{P}_S^{[1]}(t) \Big|_{t=0}, \quad k \in \mathbb{N}_0$$

- Alternatively, use FFT

Main result 2

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- We present a generating function for the cumulative allocations of risk X_1

Theorem

Let

$$\frac{\mathcal{P}_S^{[1]}(t)}{1-t} = \sum_{k=0}^{\infty} t^k E\left[X_1 \times 1_{\{S \leq k\}}\right]$$

Then, $\mathcal{P}_S^{[1]}(t)/(1-t)$ *is the OGF for the sequence of cumulative expected allocations* $\left\{E\left[X_i \times 1_{\{S \leq kh\}}\right]\right\}_{k \in \mathbb{N}_0}$.

- One can recover the values of $E\left[X_i \times 1_{\{S \leq kh\}}\right], k \in \mathbb{N}_0$ by differentiating

$$[t^k]\left\{\frac{\mathcal{P}_S^{[1]}(t)}{1-t}\right\} = E\left[X_i \times 1_{\{S \leq kh\}}\right] = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\mathcal{P}_S^{[1]}(t)}{1-t}\bigg|_{t=0}, \quad k \in \mathbb{N}_0$$

- Alternatively, use FFT

Examples – independent rvs

Independent $(a, b, 0)$ distributions

- We consider the family of $(a, b, 0)$ distributions where

$$f_M(k) = (a + b/k) f_M(k-1), \quad k \in \mathbb{N}_1$$

- Members:
  - Poisson ($a = 0, b = \lambda$)
  - Binomial ($a = -q/(1-q), b = (n+1)q/(1-q)$)
  - Negative binomial ($a = 1 - q, b = (r-1)(1-q)$)

Theorem

Let X_1 follow a $(a, b, 0)$ distribution and be independent of S_{-1}. For $|a| < 1$ and $k \in \mathbb{N}^+$, we have

$$[t^k]\mathcal{P}_S^{[1]}(t) = E\left[X_1 \times 1_{\{S=k\}}\right] = (a+b)\sum_{j=0}^{k-1} a^j f_S(k-1-j) \tag{2}$$

and

$$[t^k]\left\{\frac{\mathcal{P}_S^{[1]}(t)}{1-t}\right\} = E\left[X_1 \times 1_{\{S\leq k\}}\right] = (a+b)\sum_{j=0}^{k-1} a^j F_S(k-1-j) \tag{3a}$$

$$= (a+b)\sum_{j=0}^{k-1} \frac{1-a^{j+1}}{1-a} f_S(k-1-j). \tag{3b}$$

Independent $(a, b, 0)$ distributions

	Poisson	Negative binomial	Binomial
a	0	$1-q$	$-q/(1-q)$, for $0 < q < 1/2$
b	λ	$(r-1)(1-q)$	$(n+1)1/(1-q)$
$E[X_1 \times 1_{\{S=k\}}]$	$\lambda f_S(k-1)$	$r\sum_{j=1}^{k}(1-q)^j f_S(k-j)$	$n\sum_{j=1}^{k}(-1)^{j+1}\left(\frac{q}{1-q}\right)^j f_S(k-j)$
$E[X_1 \times 1_{\{S\leq k\}}]$ (v1)	$\lambda F_S(k-1)$	$r\sum_{j=1}^{k}(1-q)^j F_S(k-j)$	$n\sum_{j=1}^{k}(-1)^{j+1}\left(\frac{q}{1-q}\right)^j F_S(k-j)$
$E[X_1 \times 1_{\{S\leq k\}}]$ (v2)	$\lambda F_S(k-1)$	$r\frac{1-q}{q}\sum_{j=1}^{k}\left(1-(1-q)^j\right) f_S(k-j)$	$-n\sum_{j=1}^{k}\frac{1-\left(-\frac{q}{1-q}\right)^{j+1}}{1+\frac{q}{1-q}} f_S(k-j)$

Independent compound $(a, b, 0)$ distributions

- Let M be a frequency rv and $B_1 \sim B_2 \sim \cdots \sim B$ be a sequence of severity rvs (with support $\mathbb{N}_0$ in this talk)
- Let X_1 is compound distributed,

$$X_1 = \begin{cases} 0, & M = 0 \\ \sum_{j=1}^{M} B_j, & M > 0 \end{cases}$$

Independent compound $(a, b, 0)$ distributions

Proposition

Let X_1 be a compound rv with frequency rv M with f_M in the $(a, b, 0)$ family of distributions with $|a| < 1$ and discrete severity rv B, with X_1 independent of $(X_2, \ldots, X_n)$. The OGF for expected allocations is

$$\mathcal{P}_S^{[1]}(t) = t\mathcal{P}'_{B_1}(t)\mathcal{P}'_{M_1}\left(\mathcal{P}_{B_1}(t)\right)\mathcal{P}_{X_2,\ldots,X_n}(t).$$

Further, if $|a\mathcal{P}_{B_1}(t)| < 1$ for all $|t| < 1$[1], then

$$\mathcal{P}_S^{[1]}(t) = t\mathcal{P}'_{B_1}(t)\frac{a+b}{1-a\mathcal{P}_{B_1}(t)}\mathcal{P}_S(t)$$

[1] generalizable

Independent compound $(a, b, 0)$ distributions

- Let X_1 be rv in the class of compound Poisson distributions
- Severity distribution is discrete with support $\mathbb{N}_0$
- We have

$$\mathcal{P}_S^{[1]}(t) = \lambda_1 t \mathcal{P}'_{B_1}(t) \mathcal{P}_M(\mathcal{P}_{B_1}(t)) \mathcal{P}_{X_2,\ldots,X_n}(t) = \lambda_1 t \mathcal{P}'_{B_1}(t) \mathcal{P}_S(t)$$

$$[t^k]\mathcal{P}_S^{[1]}(t) = E[X_1 \times 1_{\{S=k\}}] = \lambda \sum_{l=1}^{k} l f_{B_1}(l) f_S(k-l), \quad k \in \mathbb{N}_1$$

$$[t^k]\left\{\frac{\mathcal{P}_S^{[1]}(t)}{1-t}\right\} = E[X_1 \times 1_{\{S \leq k\}}] = \lambda \sum_{l=1}^{k} E\left[B_1 \times 1_{\{B_1 \leq l\}}\right] f_S(k-l), \quad k \in \mathbb{N}_1$$

$$= \lambda \sum_{l=1}^{k} l f_{B_1}(l) F_S(k-l), \quad k \in \mathbb{N}_1$$

- Discrete version of [Denuit and Robert, 2020]

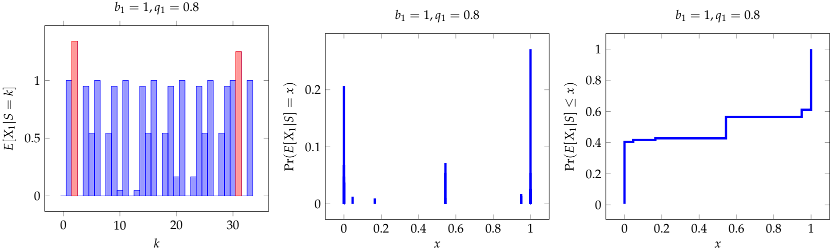
Portfolio of heterogeneous claims

- Let $\boldsymbol{I}$ be a vector of independent Bernoulli rvs with claim probability q_i
- Define $\boldsymbol{X}$ with $X_i = b_i I_i$, with $b_i \in \mathbb{N}^+$
- Computing expected allocations directly = infeasable (NP-complete) since one must find partitions of the set $\{b_1, \dots, b_n\}$
- We have

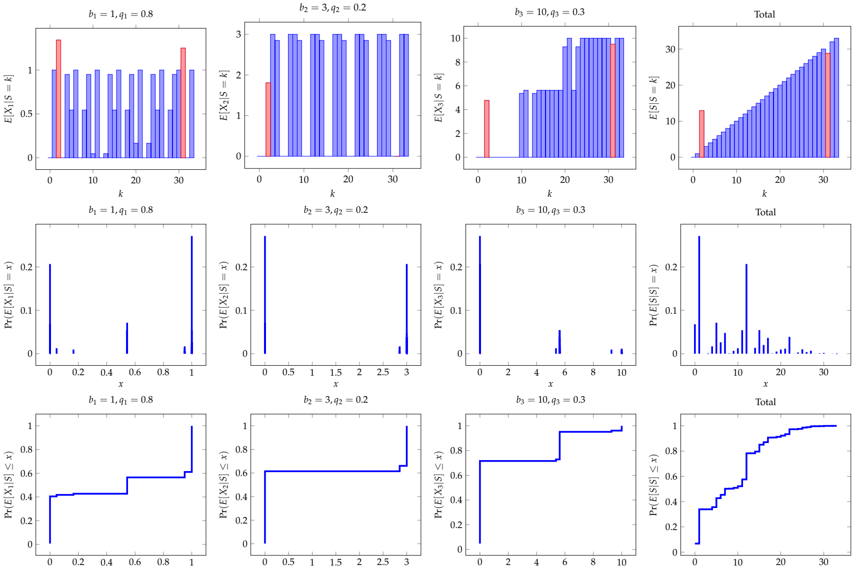
$$P_S^{[1]}(t) = q_1 b_1 t^{b_1} \prod_{i=2}^{n} (1 - q_i + q_i t^{b_i})$$

Portfolio of heterogeneous claims – example

i	1	2	3	4	5	6
b_i	1	3	10	4	5	10
q_i	0.8	0.2	0.3	0.05	0.15	0.25



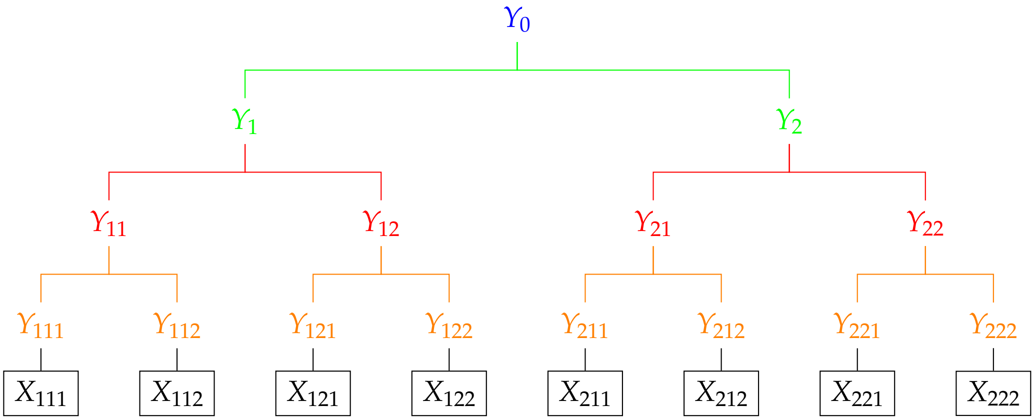
Portfolio of heterogeneous claims – example



Examples – dependent rvs

Poisson common shock

- Let $Y_A \sim Pois(\lambda_A)$
- Let $X_{ijk} = Y_{ijk} + Y_{ij} + Y_i + Y_0$ for $(i,j,k) \in \{1,2\}^3$
- Tree dependence structure



Poisson common shock

- Define

$$S = \sum_{(i,j,k) \in \{1,2\}^3} X_{ijk}$$

- Then S is a compound Poisson rv and

$$\mathcal{P}_S^{[111]}(t) = \left(\lambda_{111} t + \lambda_{11} t^2 + \lambda_1 t^4 + \lambda_0 t^8\right) \mathcal{P}_S(t)$$

- We find

$$E\left[X_{111} \times 1_{\{S=k\}}\right] = \begin{cases} 0, & k = 0 \\ \lambda_{111} f_S(k-1), & k = 1 \\ \lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2), & k = 2,3 \\ \lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2) + \lambda_1 f_S(k-4), & k = 4,\ldots,7 \\ \lambda_{111} f_S(k-1) + \lambda_{11} f_S(k-2) + \lambda_1 f_S(k-4) + \lambda_0 f_S(k-8), & k = 8,9,\ldots \end{cases}$$

Multivariate negative Binomial

- Multivariate mixed Poisson distribution
- Common mixture $\boldsymbol{\Theta} = (\Theta_1, \dots, \Theta_n)$ with $E[\Theta_i] = 1$ for $i = 1, \dots, n$
- Vector of conditionally independent rvs $(X_i | \Theta_i = \theta_i) \sim Poisson(\lambda_i \theta_i)$ for $i = 1, \dots, n$
- The pgf of $\boldsymbol{X}$ is

$$\mathcal{P}_{\mathbf{X}}(t_1, \dots, t_n) = E_{\boldsymbol{\Theta}}\left[e^{\Theta_1 \lambda_1 (t_1 - 1)} \dots e^{\Theta_n \lambda_n (t_n - 1)}\right] = \mathcal{M}_{\boldsymbol{\Theta}}(\lambda_1(t_1 - 1), \dots, \lambda_n(t_n - 1)) \tag{4}$$

- where $\mathcal{M}_{\boldsymbol{\Theta}}(t_1, \dots, t_n)$ is the moment generating function of $\boldsymbol{\Theta}$
- OGF of expected allocations:

$$\mathcal{P}_S^{[1]}(t) = \lambda_1 t \left[\frac{\partial}{\partial x} \mathcal{M}_{\boldsymbol{\Theta}}(x, \lambda_2(t-1), \dots, \lambda_n(t-1))\right]\Bigg|_{x = \lambda_1(t-1)} \tag{5}$$

Poisson-gamma common mixture

- Mixture distribution from a bivariate gamma common shock model from [Mathai and Moschopoulos, 1991]
- Define three rvs $Y_i, i \in \{0, 1, 2\}$ with
  - $Y_0 \sim Gamma(\gamma_0, \beta_0)$
  - $Y_i \sim Gamma(r_i - \gamma_0, r_i), i \in \{1, 2\}$
  - $0 \leq \gamma_0 \leq \min(r_1, r_2)$
- Let $\Theta_i = \beta_0 / r_i Y_0 + Y_i, i = 1, 2$
- (Θ_1, Θ_2) is a bivariate Gamma random vector with $\Theta_i \sim Ga(r_i, r_i), i = 1, 2$, and γ_0 is a dependence parameter
- (X_1, X_2) is a bivariate negative binomial rv

Poisson-gamma common mixture

- Moment generating functions of mixture random vector

$$\mathcal{M}_{\Theta_1,\Theta_2}(x_1, x_2) = \left(1 - \frac{x_1}{r_1}\right)^{-(r_1-\gamma_0)} \left(1 - \frac{x_2}{r_2}\right)^{-(r_2-\gamma_0)} \left(1 - \frac{x_1}{r_1} - \frac{x_2}{r_2}\right)^{-\gamma_0}$$

- PGF of S

$$\mathcal{P}_S(t) = (1 - \zeta_1(t-1))^{-(r_1-\gamma_0)} (1 - \zeta_2(t-1))^{-(r_2-\gamma_0)} (1 - \zeta_{12}(t-1))^{-\gamma_0},$$

  where $\zeta_1 = \lambda_1/r_1$, $\zeta_2 = \lambda_2/r_2$ and $\zeta_{12} = \lambda_1/r_1 + \lambda_2/r_2$
- S is the sum of three independent negative binomially distributed rvs with parameters $(r_1 - \gamma_0, 1/(1-\zeta_1))$, $(r_2 - \gamma_0, 1/(1-\zeta_2))$ and $(\gamma_0, 1/(1-\zeta_{12}))$

Poisson-gamma common mixture

- The OGF for expected allocations is

$$\mathcal{P}_S^{[1]}(t) = \lambda_1 t \left(\frac{1 - \gamma_0 / r_1}{1 - \zeta_1(t-1)} + \frac{\gamma_0 / r_1}{1 - \zeta_{12}(t-1)} \right) \mathcal{P}_S(t)$$

- Expected allocations

$$[t^k]\mathcal{P}_S^{[1]}(t) = E\left[X_1 \times 1_{\{S=k\}}\right] = \lambda_1 \sum_{j=0}^{k-1} \left[\left(1 - \frac{\gamma_0}{r_1}\right) \frac{1}{1+\zeta_1} \left(\frac{\zeta_1}{1+\zeta_1}\right)^j + \frac{\gamma_0}{r_1} \frac{1}{1+\zeta_{12}} \left(\frac{\zeta_{12}}{1+\zeta_{12}}\right)^j \right] f_S(k-1-j)$$

Fast Fourier transform method

Fast Fourier transform method

- Let $\mu_k^{[1]} = E[X_1 \times 1_{\{S=k\}}]$ for $k = 0, \ldots, k_{max} - 1$
- Let $\boldsymbol{\mu}^{[1]} = (\mu_0^{[1]}, \ldots, \mu_{k_{max}}^{[1]})$
- Discrete Fourier transform of $\boldsymbol{\mu}^{[1]}$, noted $\widehat{\boldsymbol{\mu}}^{[1]} = (\widehat{\mu}_0^{[1]}, \ldots, \widehat{\mu}_{k_{max}-1}^{[1]})$, is

$$\widehat{\mu}_j^{[1]} = \mathcal{P}_S^{[1]}\left(e^{i2\pi j/k_{max}}\right), \quad j = 0, \ldots, k_{max} - 1 \tag{6}$$

- The inverse DFT can recover the sequence of expected allocations

$$\mu_j^{[1]} = \frac{1}{k_{max}} \sum_{j=0}^{k_{max}-1} \mathrm{Re}\left(\widehat{\mu}_j^{[1]} e^{-i2\pi jk/k_{max}}\right), \quad k = 0, \ldots, k_{max} - 1$$

- See [Embrechts et al., 1993, Embrechts and Frei, 2009] for applications of FFT in actuarial science

Application 1: large portfolio

- Consider a portfolio of 10 000 risks
- Compound Poisson distributions with rate λ_j
- Severity rv $B_j \sim NBinom(r_j, q_j)$
- Simulate different parameters for each risk

$$\begin{cases} \lambda_j & \sim Exp(10) \\ r_j & \sim Unif(\{1,2,3,4,5,6\}) \\ q_j & \sim Unif([0.4, 0.5]) \end{cases}$$

- On average, $\lambda_j = 0.1$, $r_j = 3.5$ and $q_j = 0.45$
- Computes $1600 \times 10\,000$ conditional means at once
- Takes approximately 16 seconds on a personal computer

Application 1: large portfolio

Code (setup)

```
set.seed(10112021)
n_participants <- 10000
kmax <- 2^13
lam <- list(); fc <- list(); mu <- list()

lambdas <- rexp(n_participants, 10)
rs <- sample(1:6, n_participants, replace = TRUE)
qs <- runif(n_participants, 0.4, 0.5)

for(i in 1:n_participants) {
  lam[[i]] <- lambdas[i]
  fci <- dnbinom(0:(kmax-2), rs[i], qs[i])
  fc[[i]] <- c(fci, 1 - sum(fci))
}

```

Application 1: large portfolio

Code (allocations)

```
dft_fx <- list(); phic <- list(); cm <- list()

for(i in 1:n_participants) {
  dft_fx[[i]] <- exp(lam[[i]] * (fft(fc[[i]]) - 1))
  phic[[i]] <- fft(c(1:(kmax-1) * fc[[i]][-1], 0))
}

dft_fs <- Reduce("*", dft_fx)
fs <- Re(fft(dft_fs, inverse = TRUE))/kmax
e1 <- exp(-2i*pi*(0:(kmax-1))/kmax)

for(i in 1:n_participants) {
  dft_mu <- e1 * phic[[i]] * lam[[i]] * dft_fs
  mu[[i]] <- Re(fft(dft_mu, inverse = TRUE))/kmax
  cm[[i]] <- mu[[i]]/fs
}
```

Application 1: large portfolio

j	1	2	3	4	5	6	7	8
λ_j	0.16	0.03	0.03	0.24	0.12	0.47	0.15	0.01
q_j	0.49	0.42	0.46	0.45	0.49	0.44	0.44	0.48
r_j	2	6	1	4	6	5	3	1
$E[X_j]$	0.34	0.26	0.03	1.16	0.73	2.99	0.56	0.01

Table: First 8 set of parameters

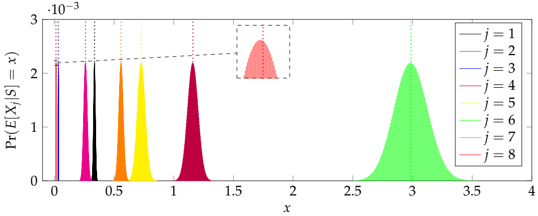


Figure: Distribution of conditional means for the first eight contracts.

Application 2

- Let $X_j, j \in \{1, \dots, n\}$: arithmetized Pareto distribution with the moment matching method
- Study $n \in \{1, 3, 100, 1\,000\}$ to observe behavior of conditional means as n increases
- Set first three risks with fixed parameters
  - Set $(\alpha_1, \alpha_2, \alpha_3) = (1.3, 1.6, 1.9)$
  - Set $\lambda_j = 10(\alpha_j - 1)$ such that $E[X_j] \approx 10, j \in \{1, 2, 3\}$
- Simulate remaining parameters
  - Simulate parameters
$$\begin{cases} \alpha_j & \sim Unif([1.3, 1.9]) \\ \lambda_j & \sim Unif([5, 15]) \end{cases}$$
  - We have $50/9 \leq E[X_j] \leq 50$ for $j \in \{4, \dots, 1000\}$
- Computes $E\left[X_i \times 1_{\{S=kh\}}\right]$ for $i \in \{1, \dots, 1\,000\}$ and $k \in \{0, \dots, 1\,048\,576\}$
- Approximately 9 minutes on a personal computer to compute 1 billion values

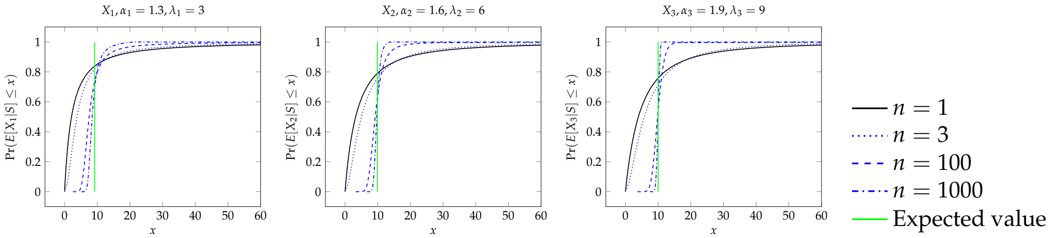


Figure: Cumulative distribution function of conditional means for $n = 3, 100, 1000$.

Conclusion

- Propose new method to compute expected allocations
- Practical applications for peer-to-peer insurance and capital allocation
- Convenient results for independent compound $(a, b, 0)$ distributions
- Convenient results for dependent rvs (is the pgf easy to differentiate ?)
- Efficient algorithm using FFT, even with large heterogeneous portfolios of heavy-tailed risks

Conclusion

Thanks for your attention!

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