Stochastic representation of FGM copulas using multivariate Bernoulli random variables

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Abstract

We establish a one-to-one correspondence between Fréchet's class of multivariate Bernoulli distribution with symmetric marginals and the well-known family of FGM copulas. We introduce a new stochastic representation of the family of d-variate FGM copulas. The representation is bijective: we show that from any d-variate Bernoulli distribution, we define a corresponding d-variate FGM copula; and we show that for any d-variate FGM copula, we find the corresponding d-variate Bernoulli distribution. The proposed stochastic representation has many advantages, notably establishing stochastic orders, constructing subclasses of FGM copulas and sampling. In particular, we use the stochastic representation to develop computational methods to perform sampling from subclasses of FGM copulas, which scale well to large dimensions.

Keywords: Multivariate Farlie-Gumbel-Morgenstern copulas, Multivariate Bernoulli distributions, Stochastic representation, Stochastic simulation, Dependence ordering

1 Introduction

A copula is a multivariate distribution function with standard uniform margins. An important family of copulas is the Farlie-Gumbel-Morgenstern (FGM) class of copula, first studied by Eyraud (1936), Farlie (1960), Gumbel (1960) and Morgenstern (1956). Standard references for FGM copulas are Cambanis (1977), Johnson and Kott (1975), (Kotz and Drouet, 2001, Chapter 5), (Kotz et al., 2004, Section 44.10). Often, one describes the FGM copula as a perturbation of the product copula (see, e.g. Durante and Sempi (2015)), inducing moderate dependence between margins. Also, the author of Nelsen (2007) indicates the FGM copula is a first-order approximation to the Ali-Mikhail-Haq (AMH), Frank and Placket copulas.

The class of FGM copulas is a popular choice when working in two dimensions distributions due to its simple shape and the exact calculus of polynomial functions. Examples of applications of FGM copulas include finance Mai and Scherer (2014), actuarial science Bargès et al. (2011) and bioinformatics Kim et al. (2008). When increasing dimensions, the copula retains polynomial shape, but loses practical interest since the admissible parameter space becomes complex. For this reason, the properties of FGM copulas in large dimensions are largely unknown.

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This paper uncovers a stochastic representation of FGM copulas using symmetric multivariate Bernoulli distributions. This relationship allows us to leverage the extensive literature on multivariate Bernoulli distributions to uncover new results for the family of FGM copulas. The discovery of stochastic representations of multivariate models often generates new knowledge and interest in the particular model. For instance, McNeil and Nešlehová (2009) propose a representation of Archimedean copula generators using the Williamson d-transform, leading to new Archimedean copula families and efficient sampling procedures. Using the main result of the current paper, one can also develop a new understanding of FGM copulas because symmetric multivariate Bernoulli distributions turn out to be central to the construction of FGM distributions. The proposed stochastic representation makes it feasible to use FGM copulas for high-dimensional problems. By conditioning on the latent Bernoulli random variable, one can compute statistics under independence, replacing tedious mathematical formulas with many computations. As an example, we propose a stochastic sampling method that scales well to high dimensions.

We organize the paper as follows. Section 2 presents notation, definitions and basic results on copula theory and FGM distributions. The main result of this paper appears in Section 3, in which we show that exponential FGM distributions (and therefore FGM copulas) have a stochastic representation based on symmetric multivariate Bernoulli distributions. In Section 4, we provide new results on dependence ordering within the class of FGM copulas, also identifying the extremal positive dependence FGM copula under different stochastic orderings. We leverage these findings to provide results for measures of multivariate association in Section 5. Then, Sections 6 and 7 explore in-depth bivariate and trivariate FGM distributions. In Section 8, we provide an example of how one can use a known symmetric multivariate Bernoulli distribution to construct a subfamily of FGM distributions. Based on a Markov-Bernoulli process, we construct a subfamily of FGM copulas exhibiting autoregressive dependence structures between marginals. Section 9 provides a new algorithm to sample from multivariate FGM distributions that scale well to large dimensions.

2 FGM copulas and basic results

This section provides background on copulas and FGM distributions. We begin by presenting notation. Let $\boldsymbol{x}$ denote a vector $(x_1, \dots, x_d) \in \mathbb{R}^d$. All expressions such as $\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} \times \boldsymbol{y}$ or $\boldsymbol{x} \leq \boldsymbol{y}$ represent componentwise operations. The symbol $\boldsymbol{X}$ is reserved for a random vector on $\mathbb{R}^d$ with cumulative distribution function (cdf) $F_{\boldsymbol{X}}$ defined by $F_{\boldsymbol{X}}(\boldsymbol{x}) = \Pr(\boldsymbol{X} \leq \boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^d$. We denote the Fréchet class of univariate marginals $F_1, \dots, F_d$ by $\Gamma(F_1, \dots, F_d)$.

Let $B(x)$ be the cdf of a symmetric Bernoulli distribution, that is, $B(x) = F_I(x) = \frac{1}{2}1_{[0,\infty)}(x) + \frac{1}{2}1_{[1,\infty)}(x)$, $x \geq 0$, where $1_A(x) = 1$, if $x \in A$, and $1_A(x) = 0$, otherwise. The multivariate rv $\boldsymbol{I}$ represents a d-dimensional vector of binary rvs with joint cdf $F_{\boldsymbol{I}} \in \mathcal{B}_d$, where $\mathcal{B}_d = \Gamma(B_1, \dots, B_d)$. The joint probability mass function (pmf) of $\boldsymbol{I}$ is denoted by $f_{\boldsymbol{I}}$ where $f_{\boldsymbol{I}}(\boldsymbol{i}) = \Pr(I_1 = i_1, \dots, I_d = i_d)$, for $\boldsymbol{i} = (i_1, \dots, i_d) \in \{0,1\}^d$.

Let $\mathcal{E}_d$ denote the Fréchet class of all multivariate exponential distributions with exponential of mean 1 as marginals, whose cdfs are $G_j(x) = G(x) = 1 - e^{-x}$, $x \geq 0$, for $j \in \{1, ..., d\}$, that is, $\mathcal{E}_d = \Gamma(G_1, ..., G_d)$.

Definition 2.1. A (d-variate) copula is a function $C : [0,1]^d \to [0,1]$ satisfying

1. $C(u_1, \dots, u_d) = 0$ if any $u_j = 0$, $j \in \{1, \dots, d\}$.

2. $C(u_1, \dots, u_d) = u_j$ if $u_k = 1$ for all $k \in \{1, \dots, d\}$ and $k \neq j$.

3. C is d-increasing on $[0,1]^d$, that is,

$$\sum_{i_1=1}^{2}\cdots\sum_{i_d=1}^{2}(-1)^{i_1+\cdots+i_d}C(u_{1i_1},\ldots,u_{di_d})\geq 0,$$

for all $0\leq u_{j1}\leq u_{j2}\leq 1$ and $j\in\{1,\ldots,d\}$.

We denote $\mathcal{C}_d$ as the class of d-variate copulas. The following Theorem from Sklar (1959) is the essential tool to extract copulas from multivariate distributions.

Theorem 2.2 (Sklar's Theorem)**.** *Let H be a d-variate distribution function with margins $F_1,\ldots,F_d$. Then there exists a d-variate copula C such that for all $\boldsymbol{x}\in\mathbb{R}^d$,*

$$H(x_1,\ldots,x_d)=C(F_1(x_1),\ldots,F_d(x_d)).$$

If H has continuous margins, the copula C is unique, otherwise it is unique on the set

$$\mathit{Range}(F_1)\times\cdots\times\mathit{Range}(F_d).$$

In particular, if $\boldsymbol{X}$ is a continuous random vector with cdf $F_{\boldsymbol{X}}$ and margins $F_1,\ldots,F_d$ and $\boldsymbol{U}$ is a random vector distributed as the copula C, we have $\boldsymbol{U}\overset{d}{=}(F_1(X_1),\ldots,F_d(X_d))$ and $\boldsymbol{X}\overset{d}{=}(F_1^{-1}(U_1),\ldots,F_d^{-1}(U_d))$, where $\overset{d}{=}$ means equality in distribution.

2.1 Family of exponential FGM distributions

We define $\mathcal{E}_d^{FGM}$ as the family of all d-variate FGM distributions with exponential of mean 1 as marginals, with $\mathcal{E}_d^{FGM}\subset\mathcal{E}_d$. A d-variate exponential FGM distribution has cdf $F\in\mathcal{E}_d^{FGM}$

$$F(\boldsymbol{x})=\prod_{m=1}^{d}\left(1-e^{-x_m}\right)\left(1+\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\prod_{l=1}^{k}e^{-x_{j_l}}\right),\quad \boldsymbol{x}\in\mathbb{R}_+^d, \tag{1}$$

with parameters denoted by the vector

$$\boldsymbol{\theta}=\left(\theta_{j_1\ldots j_k}:1\leq j_1<\cdots<j_k\leq d,\ k\in\{2,\ldots,d\}\right). \tag{2}$$

We call $\boldsymbol{\theta}$ the dependence parameter vector since it captures the dependence induced in the multivariate distribution. In addition, when referring to the parameters with k indices $\theta_{j_1\ldots j_k}$, $1\leq j_1<\cdots<j_k\leq d$, we use the term $k-$dependence parameters, $k\in\{2,\ldots,d\}$. The number of parameters in a d-variate exponential FGM distribution is

$$d^\star=|\boldsymbol{\theta}|=2^d-d-1. \tag{3}$$

The d-variate exponential FGM distribution exists if $\boldsymbol{\theta}\in\mathcal{T}_d$ where

$$\mathcal{T}_d=\left\{\boldsymbol{\theta}\in\mathbb{R}^{d^\star}:1+\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\varepsilon_{j_1}\varepsilon_{j_2}\ldots\varepsilon_{j_k}\geq 0\right\}, \tag{4}$$

for $\{\varepsilon_{j_1},\varepsilon_{j_2},\ldots,\varepsilon_{j_k}\}\in\{-1,1\}^d$. The 2^d constraints of its definition in (4), which has been derived in Cambanis (1977), imply that $\mathcal{T}_d\subseteq[-1,1]^{d^\star}$. Finally, each k-margin of $F\in\mathcal{E}_d^{FGM}$ is a multivariate exponential distribution in $\mathcal{E}_k^{FGM}$ for $k\in\{2,\ldots,d\}$.

2.2 Family of FGM copulas

Let $\mathcal{C}_d^{FGM}$ be the subset of $\mathcal{C}_d$ corresponding to the family of d-variate FGM copulas. One obtains the family of FGM copulas by evoking Sklar's Theorem on exponential FGM distributions. Since $F \in \mathcal{E}_d^{FGM}$, the copula C associated to F is obtained applying the inversion method described in Section 3.1 of Nelsen (2007). Denoting $F_{Y_j}^{-1}$ as the generalized inverse of F_{Y_j}, that is $F_{Y_j}^{-1}(u) = \inf\{x \in \mathbb{R} : F_{Y_j}^{-1}(x) \geq u\}$ for $u \in [0,1]$ and $j \in \{1, ..., d\}$, we find

$$C(u_1, \ldots, u_d) = F_{\boldsymbol{Y}}\left(F_{Y_1}^{-1}(u_1), \ldots, F_{Y_d}^{-1}(u_d)\right), \quad \boldsymbol{u} \in [0,1]^d,$$

where $F_{Y_j}^{-1}(u) = G^{-1}(u) = -\ln(1-u)$, $u \in [0,1]$ and $\lim_{u \uparrow 1} G^{-1}(u) = \infty$. One obtains the expression for a FGM copula:

$$C(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(1 + \sum_{k=2}^{d} \sum_{1 \leq j_1 < \cdots < j_k \leq d} \theta_{j_1 \ldots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \ldots \overline{u}_{j_k}\right) \quad \boldsymbol{u} \in [0,1]^d, \tag{5}$$

where $\overline{u}_j = 1 - u_j$, $j \in \{1, \ldots, d\}$. When $d = 2$, (5) becomes the well-known expression of bivariate FGM copulas with $d^\star = 1$ parameter (denoted θ_{12}) and given by

$$C(u_1, u_2) = u_1 u_2 + \theta_{12} u_1 u_2 \overline{u}_1 \overline{u}_2, \quad (u_1, u_2) \in [0,1]^2,$$

with $\theta_{12} \in \mathcal{T}_2 = [-1, 1]$. The association measures Kendall's tau and Spearman's rho are $\tau = 2\theta_{12}/9$ and $\rho = \theta_{12}/3$ respectively.

3 Main result

The stochastic representation of a d-variate FGM copula is obtained through the joint cdf of the vector of rvs defined as follows. Let $\boldsymbol{I}$ be a vector of multivariate Bernoulli rvs, $\boldsymbol{Z}_0 = (Z_{1,0}, \ldots, Z_{d,0})$ be a vector of independent exponential rvs with mean $1/2$ and $\boldsymbol{Z}_1 = (Z_{1,1}, \ldots, Z_{d,1})$ be a vector of independent exponential rvs with mean 1.

Theorem 3.1. *Define the vector of rvs* $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ *as*

$$Y_j = Z_{j,0} + I_j Z_{j,1}, \quad \in \{1, \ldots, d\}. \tag{6}$$

Then, $\boldsymbol{Y}$ *follows a multivariate distribution with exponential marginals with mean 1 and joint cdf given by*

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \sum_{\boldsymbol{i} \in \{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{j=1}^{d} \left(1 - e^{-x_j}\right)\left(1 + (-1)^{i_j} e^{-x_j}\right), \quad \boldsymbol{x} \in \mathbb{R}_+^d. \tag{7}$$

In the following theorem, we identify the dependence structure behind $F_{\boldsymbol{Y}} \in \mathcal{E}_d^{FGM}$.

Theorem 3.2. *Let* $\boldsymbol{I}$ *be a vector of Bernoulli rvs with* $F_{\boldsymbol{I}} \in \mathcal{B}_d$, $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ *with* $Y_j, j \in \{1, \ldots, d\}$ *as defined in* (6). *Then,* (1) *and* (7) *both uniquely define an exponential FGM distribution, with*

$$\theta_{j_1 \ldots j_k} = (-2)^k E_{\boldsymbol{I}}\left[\prod_{l=1}^{k}\left(I_{j_l} - \frac{1}{2}\right)\right]. \tag{8}$$

Indeed, the following bijection holds:

- *Let $\boldsymbol{\theta} \in \mathcal{T}_d$, then there exists a rv $\boldsymbol{I}$ with $F_{\boldsymbol{I}} \in \mathcal{B}_d$ with pmf*

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \frac{1}{2^d}\left(1 + \sum_{k=2}^{d} \sum_{1 \leq j_1 < \cdots < j_k \leq d} (-1)^{i_{j_1}+\cdots+i_{j_k}}\, \theta_{j_1 \ldots j_k}\right), \quad \boldsymbol{i} \in \{0,1\}^d. \tag{9}$$

- *Let $F_{\boldsymbol{I}} \in \mathcal{B}_d$, then there exists a $F_{\boldsymbol{Y}} \in \mathcal{E}_d^{FGM}$ with*

$$\theta_{j_1 \ldots j_k} = \sum_{(i_{j_1},\ldots,i_{j_k}) \in \{0,1\}^k} (-1)^{i_{j_1}+\cdots+i_{j_k}}\, f_{I_{j_1},\ldots,I_{j_k}}\left(i_{j_1},\ldots,i_{j_k}\right), \tag{10}$$

for $1 \leq j_1 < \cdots < j_k \leq d$ and $k \in \{2,\ldots,d\}$.

From the expression $F_{\boldsymbol{Y}}$ in Theorem 3.1, we propose an alternative representation of a d-variate FGM copula in the following corollary.

Corollary 3.3. *If the conditions of Theorem 3.1 are satisfied, then the copula C that is directly extracted from the expression of $F_{\boldsymbol{Y}}$ in (7) has the following form:*

$$C(\boldsymbol{u}) = \sum_{\boldsymbol{i} \in \{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{m=1}^{d} u_m\left(1 + (-1)^{i_m}\overline{u}_m\right), \quad \boldsymbol{u} \in [0,1]^d. \tag{11}$$

Let us examine the values of $\boldsymbol{\theta}$ in (8). Following Chapter 34, Section 2.1 in Johnson et al. (1997), we denote the $\boldsymbol{r} = (r_1,\ldots,r_d)$'th central mixed moment of $\boldsymbol{I}$ by

$$\mu_{\boldsymbol{r}}(\boldsymbol{I}) = E\left[\prod_{j=1}^{d}(I_j - E[I_j])^{r_j}\right], \tag{12}$$

where $\boldsymbol{r} \in \mathbb{N}^d$. Using (12), the expression for the parameter $\theta_{j_1\ldots j_k}$ in (8) becomes

$$\theta_{j_1\ldots j_k} = (-2)^k \mu_{\boldsymbol{1}_k}(I_{j_1},\ldots,I_{j_k}),$$

where $\boldsymbol{1}_k$ is a k-dimensional vector of ones, $1 \leq j_1 < \cdots < j_k \leq d$, and $k \in \{2,\ldots,d\}$. Note that one can construct a multivariate Bernoulli distribution by specifying every probability $f_{\boldsymbol{I}}(\boldsymbol{i}), \boldsymbol{i} \in \{0,1\}^d$, or by specifying the central moments in (12), see Teugels (1990) for some equivalence formulas. A FGM copula is therefore defined, up to a constant, by the central mixed moment of the latent Bernoulli rvs. From here on, we call the representation in (5) with dependence parameter vector $\boldsymbol{\theta}$, the *natural representation*, due to the link between the dependence parameter vector $\boldsymbol{\theta}$ and the central moments of the Bernoulli distribution $(-2)^k\mu_{\boldsymbol{1}_k}(I_{j_1},\ldots,I_{j_k})$, $1 \leq j_1 < \cdots < j_k \leq d$, and $k \in \{2,\ldots,d\}$. We call the representation in (11) the *stochastic representation*.

Remark 3.4. The expression of C in (11) allows one to establish a link to another stochastic representation of a FGM copula. Let $\boldsymbol{V}_j = (V_{1,j},\ldots,V_{d,j})$, $j \in \{0,1\}$, be a vector of independent and identically distributed (iid) rvs, where $V \sim Unif(0,1)$. Define $V_{m,[0]} = \min(V_{m,0}, V_{m,1})$ and $V_{m,[1]} = \max(V_{m,0}, V_{m,1})$, $m \in \{1,\ldots,d\}$. Since $V_{m,[i_m]} \sim Beta(1+i_m, 2-i_m)$, we have $F_{V_{m,[i_m]}}(u_m) = (u_m^2)^{i_m}(1-(1-u_m)^2)^{1-i_m} = u_m\left(1+(-1)^{i_m}\overline{u}_m\right)$, $u_m \in [0,1]$, $m \in \{1,\ldots,d\}$. Then, C in (11) is also

$$C(\boldsymbol{u}) = \sum_{\boldsymbol{i} \in \{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{m=1}^{d} F_{V_{j,[i_m]}}(u_m) = E_{\boldsymbol{I}}\left[\prod_{m=1}^{d} F_{V_{j,[I_m]}}(u_m)\right], \quad \boldsymbol{u} \in [0,1]^d. \tag{13}$$

Hence, we establish a connection between the novel stochastic representation and the order-statistics-based method to generate a FGM copula described in Baker (2008). In the current paper, we go further by studying the underlying multivariate Bernoulli distribution in connection with the dependence parameter vector $\boldsymbol{\theta}$. Also, (11) and (13) emphasize the link between our stochastic representation and the empirical beta copula (with $n = 2$) studied in Segers et al. (2017).

Remark 3.5. If one does not impose any structure on the natural representation, the number of parameters of a d-variate FGM copula is $d^\star = |\boldsymbol{\theta}| = 2^d - d - 1$. For $d = 2, 5, 10, 20$, the number of parameters grows at an exponential rate to $d^\star = 1, 26, 1\ 013, 1\ 048\ 555$, which hinders the use of FGM copulas in practical contexts even when d is not so large. One can build on Theorem 3.2's conclusions to propose subfamilies of FGM copulas with solely one parameter in which the estimation of the parameters becomes tractable, see Section 8 for an example. Theorem 3.2 certifies the validity of parameters $\boldsymbol{\theta} \in \mathcal{T}_d$ when one defines a $F_{\boldsymbol{I}} \in \mathcal{B}_d$ and uses (10).

Remark 3.6. Another issue with the natural representation is parameter interpretation. The parameters of the d-variate FGM copula becomes difficult to interpret when d becomes large. Even for simple parameters, it isn't obvious how parameters relate from one dimension to the next. For instance, we will show that for $d = 2$, extreme positive dependence is achieved when $\theta_{12} = 1$. For $d = 3$, extreme positive dependence is achieved for $\theta_{12} = \theta_{13} = \theta_{23} = 1$ and $\theta_{123} = 0$. Hence, one could believe that the pattern $\theta_{j_1 j_2}, 1 \leq j_1 < j_2 \leq d$ and k-dependence parameters equal to zero for $k \in \{3, \ldots, d\}$ yields a valid copula. However, when $d = 4$ with the set of parameters $\boldsymbol{\theta} = (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$, evaluating the copula density function at $\boldsymbol{u} = (0, 0, 1, 1)$ yields -1, so the set of parameters does not yield a valid copula according to Definition 2.1. However, one can leverage the stochastic representation along with the Fréchet-Hoeffding upper bound of multivariate Bernoulli distributions to determine the shape of the copula under extreme positive dependence. The representation from Theorem 3.2 provides an interpretation after converting the dependence parameters into the pmf of $\boldsymbol{I}$, see Section 4.2 for more on the extreme positive dependence of FGM copulas.

Other researchers have found a link between multivariate Bernoulli rvs and the family of FGM copulas. In Sharakhmetov and Ibragimov (2002) and Fontana and Semeraro (2018), the authors show that the pmf of multivariate Bernoulli rvs with given means can be expressed as a polynomial representation that has the shape of a FGM copula. In this paper, we show that any FGM copula has a stochastic representation that has a one-to-one relationship with symmetric multivariate Bernoulli rvs. In the remainder of this paper, we investigate the implications of the main result.

4 Dependence ordering

We aim to compare vectors of rvs, say $\boldsymbol{V} = (V_1, \ldots, V_d)$ and $\boldsymbol{V}' = (V_1', \ldots, V_d')$, where, for each $j \in \{1, \ldots, d\}$, V_j and V_j' have the same marginal distribution. Given this condition on the marginals, we rely on dependence stochastic orders. We refer the reader to Sections 3.8 and 3.9 of Müller and Stoyan (2002) for a detailed presentation of the topics briefly recalled in this section, in particular the supermodular order and three other dependence orders: the lower concordance order ($\preceq_{cL}$), the upper concordance order ($\preceq_{cU}$), and the concordance order ($\preceq_c$). According to Definition 3.8.5 of Müller and Stoyan (2002), $\boldsymbol{V} \preceq_{cL} \boldsymbol{V}'$ if $F_{\boldsymbol{V}}(\boldsymbol{x}) \leq F_{\boldsymbol{V}'}(\boldsymbol{x})$ for all $\boldsymbol{x}$ and $\boldsymbol{V} \preceq_{cU} \boldsymbol{V}'$ if $\overline{F}_{\boldsymbol{V}}(\boldsymbol{x}) \leq \overline{F}_{\boldsymbol{V}'}(\boldsymbol{x})$ for all $\boldsymbol{x}$. If $\boldsymbol{V} \preceq_{cL} \boldsymbol{V}'$ and $\boldsymbol{V} \preceq_{cU} \boldsymbol{V}'$, then $\boldsymbol{V} \preceq_c \boldsymbol{V}'$.

Definition 4.1 (Supermodular order). We say $\boldsymbol{V}$ is smaller than $\boldsymbol{V}'$ under the supermodular order, denoted $\boldsymbol{V} \preceq_{sm} \boldsymbol{V}'$, if $E\left[\phi(\boldsymbol{V})\right] \leq E\left[\phi(\boldsymbol{V}')\right]$ for all supermodular functions ϕ, given that the expectations exist. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if

$$\begin{aligned} & \phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_d) - \phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_d) \\ \geq \quad & \phi(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_d) - \phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) \end{aligned}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^d$, $1 \leq i \leq j \leq d$ and all ε, $\delta > 0$.

Additional details on supermodular order can be found in Shaked and Shanthikumar (2007) and Denuit et al. (2006). Note that the supermodular order satisfies the nine desired properties for dependence orders, as mentioned in Section 3.8 of Müller and Stoyan (2002). As shown in Theorem 3.9.5 and Example 3.9.7 of Müller and Stoyan (2002), the supermodular order is stronger than the concordance order. For that reason, we focus mainly on the supermodular order.

4.1 General result

Let $\boldsymbol{I}$ and $\boldsymbol{I}'$ be multivariate symmetric Bernoulli distributions. Let $\boldsymbol{Z}_0'$, and $\boldsymbol{Z}_1'$ be copies of $\boldsymbol{Z}_0$, and $\boldsymbol{Z}_1$, where $\boldsymbol{I}, \boldsymbol{Z}_0$, $\boldsymbol{Z}_1$, $\boldsymbol{I}', \boldsymbol{Z}_0'$, and $\boldsymbol{Z}_1'$ are independent. From (6), we define $\boldsymbol{Y} = \boldsymbol{Z}_0 + \boldsymbol{I}\boldsymbol{Z}_1$ and $\boldsymbol{Y}' = \boldsymbol{Z}_0' + \boldsymbol{I}'\boldsymbol{Z}_1'$. By Theorem 3.2, it follows that $F_{\boldsymbol{Y}}, F_{\boldsymbol{Y}'} \in \mathcal{E}_d^{FGM}$. Define $\boldsymbol{U} = (U_1, \ldots, U_d)$ and $\boldsymbol{U}' = (U_1', \ldots, U_d')$ with

$$U_j = 1 - \exp(-Y_j) \quad \text{and} \quad U_j' = 1 - \exp(-Y_j'), \quad j \in \{1, \ldots, d\}. \tag{14}$$

Then, by Theorem 3.2, $F_{\boldsymbol{U}} = C$ and $F_{\boldsymbol{U}'} = C'$ with $C, C' \in \mathcal{C}_d^{FGM}$. Finally, we define the two vectors of rvs $\boldsymbol{X} = (X_1, ..., X_d)$ and $\boldsymbol{X}' = (X_1', ..., X_d')$ where $F_{X_j} = F_{X_j'} = H_j$, $j \in \{1, ..., d\}$, and

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C(H_1(x_1), ..., H_d(x_d)) \quad \text{and} \quad F_{\boldsymbol{X}'}(\boldsymbol{x}) = C'(H_1(x_1), ..., H_d(x_d)), \tag{15}$$

for $\boldsymbol{x} \in \mathbb{R}^d$. Let $\mathcal{H}_d = \Gamma(H_1, ..., H_d)$ be the Fréchet class of all multivariate distributions with univariate marginals $H_1, \ldots, H_d$. We define $\mathcal{H}_d^{FGM} \subset \mathcal{H}_d$ as the family of all multivariate FGM distributions defined with copula $C \in \mathcal{C}_d^{FGM}$ and univariate marginals $H_1, \ldots, H_d$, as in (15). It follows that

$$X_j \stackrel{d}{=} H_j^{-1}(U_j) \quad \text{and} \quad X_j' \stackrel{d}{=} H_j^{-1}(U_j'), \quad j \in \{1, ..., d\}. \tag{16}$$

Now, we are in a position to state the following result.

Theorem 4.2. *The following relationships hold:*

1. *If $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$ holds, then $\boldsymbol{Y} \preceq_{sm} \boldsymbol{Y}'$, $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$, and $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}'$.*
2. *If $\boldsymbol{I} \preceq_{c} \boldsymbol{I}'$ holds, then $\boldsymbol{Y} \preceq_{c} \boldsymbol{Y}'$, $\boldsymbol{U} \preceq_{c} \boldsymbol{U}'$, and $\boldsymbol{X} \preceq_{c} \boldsymbol{X}'$.*
3. *If $\boldsymbol{I} \preceq_{cU} \boldsymbol{I}'$ holds, then $\boldsymbol{Y} \preceq_{cU} \boldsymbol{Y}'$, $\boldsymbol{U} \preceq_{cU} \boldsymbol{U}'$, and $\boldsymbol{X} \preceq_{cU} \boldsymbol{X}'$.*
4. *If $\boldsymbol{I} \preceq_{cL} \boldsymbol{I}'$ holds, then $\boldsymbol{Y} \preceq_{cL} \boldsymbol{Y}'$, $\boldsymbol{U} \preceq_{cL} \boldsymbol{U}'$, and $\boldsymbol{X} \preceq_{cL} \boldsymbol{X}'$.*

Proof. We provide a detailed proof of item 1; the three other items follows the same sequence of arguments. The proof follows from theorems which can be found in Shaked and Shanthikumar (2007); the references to every theorem in this proof refer to the latter source. Applying both Theorems 9.A.14 and 9.A.9(b) with (7) and $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$, it follows that $\boldsymbol{Y} \preceq_{sm} \boldsymbol{Y}'$. Since the supermodular order is closed under all increasing transformations as stated in Theorem 9.A.9(a), $\boldsymbol{Y} \preceq_{sm} \boldsymbol{Y}'$ and (14) implies $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$. Using the same argument, $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}'$ follows by combining $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$ and (16).

For the concordant case, by Theorems 9.A.6, 9.A.5(a), (7), and $\boldsymbol{I} \preceq_c \boldsymbol{I}'$, we obtain $\boldsymbol{Y} \preceq_c \boldsymbol{Y}'$. Since the concordance order is closed under all increasing transformations as stated in the comments of Theorem 9.A.4, we have $\boldsymbol{U} \preceq_c \boldsymbol{U}'$ and $\boldsymbol{X} \preceq_c \boldsymbol{X}'$.

The lower and upper orthant cases follow from Theorems 6.G.7, 6,G3(b), (7), and $\boldsymbol{I} \preceq_{cL} [\preceq_{cU}]\, \boldsymbol{I}'$, we obtain $\boldsymbol{Y} \preceq_{cL} [\preceq_{cU}]\boldsymbol{Y}'$. Since the lower [upper] concordance order is closed under all increasing transformations as stated in Theorem 6.G.3(a), we have $\boldsymbol{U} \preceq_{cL} [\preceq_{cU}]\, \boldsymbol{U}'$ and $\boldsymbol{X} \preceq_{cL} [\preceq_{cU}]\, \boldsymbol{X}'$. $\square$

Theorems 3.2 and 4.2 enable one to derive the conditions under which the supermodular (or dependence) order holds within $\mathcal{E}_d^{FGM}$, $\mathcal{C}_d^{FGM}$, and $\mathcal{H}_d^{FGM}$, given that the corresponding conditions are satisfied such that the supermodular and the three other dependence orders hold within $\mathcal{B}_d$. Result 1 of Theorem 4.2 has important implications for risk modeling in finance and actuarial science, as exposed in Denuit et al. (2006).

4.2 Extremal positive dependent element

Theorems 3.2 and 4.2 provide the tools to identify extremal positive dependent element on $\mathcal{C}_d^{FGM} \subset \mathcal{C}_d$ and $\mathcal{H}_d^{FGM} \subset \mathcal{H}_d$, under the supermodular order, for any $d \geq 2$. The inequality

$$C(\boldsymbol{u}) \leq M(\boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d,$$

holds for all $C \in \mathcal{C}_d$, where M is the Fréchet-Hoeffding upper bound copula defined by

$$M(\boldsymbol{u}) = \min(u_1, \ldots, u_d), \quad \boldsymbol{u} \in [0,1]^d. \tag{17}$$

As mentioned, for example, in Section 4.7.4.1 of Denuit et al. (2006), the Fréchet-Hoeffding upper bound copula M as defined in (17) is the cdf of the vector of comonotonic rvs $\boldsymbol{U}^+ = (U_1^+, ..., U_d^+)$, that is $F_{\boldsymbol{U}^+} = M$.

Let the rv W follow a standard uniform distribution, that is, $W \sim Unif(0,1)$. The two vectors of comonotonic rvs $\boldsymbol{I}^+ = (I_1^+, ..., I_d^+)$ and $\boldsymbol{X}^+ = (X_1^+, ..., X_d^+)$ are represented in terms of this single rv with $I_j^+ \overset{d}{=} B^{-1}(W)$ and $X_j^+ \overset{d}{=} H^{-1}(W)$, $j \in \{1, \ldots, d\}$. The cdfs of the two vectors of comonotonic rvs $\boldsymbol{I}^+$ and $\boldsymbol{X}^+$ correspond to

$$F_{\boldsymbol{I}^+}(\boldsymbol{i}) = M(B_1(i_1), \ldots, B_d(i_d)), \quad \boldsymbol{i} \in \{0,1\}^d,$$

and

$$F_{\boldsymbol{X}^+}(\boldsymbol{x}) = M(H_1(x_1), \ldots, H_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Note that the values of the joint pmf of all subsets $(I_{j_1}^+, \ldots, I_{j_k}^+)$ of $\boldsymbol{I}^+$ are given by

$$f_{I_{j_1}^+, \ldots, I_{j_k}^+}(i_{j_1}, \ldots, i_{j_k}) = \begin{cases} \frac{1}{2}, & i_{j_1} = \cdots = i_{j_k} = 0, \\ \frac{1}{2}, & i_{j_1} = \cdots = i_{j_k} = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

for $1 \leq j_1 < j_2 < \ldots < j_k \leq d$ and $k \in \{2, \ldots, d\}$.

By Proposition 6.3.6 of Denuit et al. (2006), the following relations hold:

$$\boldsymbol{I} \preceq \boldsymbol{I}^+ \ \forall\ F_{\boldsymbol{I}} \in \mathcal{B}_d, \tag{19}$$

$\boldsymbol{U} \preceq \boldsymbol{U}^+ \ \forall\ C \in \mathcal{C}_d$, and $\boldsymbol{X} \preceq \boldsymbol{X}^+ \ \forall\ F_X \in \mathcal{H}_d$. While $F_{\boldsymbol{I}^+} \in \mathcal{B}_d$, we know that $F_{\boldsymbol{U}^+} = M \notin \mathcal{C}_d^{FGM}$ and $F_{\boldsymbol{X}^+} \notin \mathcal{H}_d^{FGM}$. In other words, the Fréchet-Hoeffding upper bound for $\boldsymbol{I}$ does not induce the Fréchet-Hoeffding upper bound for $\boldsymbol{U}$ or $\boldsymbol{X}$.

However, by Theorem 4.2 and the one-to-one correspondence, we know that, under the supermodular order sense (referring to the expression proposed in section 6.3.7 of Denuit et al. (2006)), there is a unique extremal positive dependent (EPD) element $\boldsymbol{U}^{EPD}$ with $F_{\boldsymbol{U}^{EPD}} = C^{EPD} \in \mathcal{C}_d^{FGM}$ such that $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}^{EPD}$ holds for all $F_{\boldsymbol{U}} = C \in \mathcal{C}_d^{FGM}$. Similarly, by (15), there is a unique extremal positive dependent vector of rvs $\boldsymbol{X}^{EPD}$ with $F_{\boldsymbol{X}^{EPD}} \in \mathcal{H}_d^{FGM}$, where $F_{\boldsymbol{X}^{EPD}}(\boldsymbol{x}) = C^{EPD}\left(H_1(x_1), ..., H_d(x_d)\right)$ for $\boldsymbol{x} \in \mathbb{R}^d$, such that $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}^{EPD}$ holds for all $F_{\boldsymbol{X}} \in \mathcal{H}_d^{FGM}$.

In the following, we identify the extremal positive dependent element of $\mathcal{C}_d^{FGM}$ under the supermodular order sense, and we provide the expression for the EPD copula C^{EPD} within the family of multivariate FGM copulas.

Theorem 4.3. *A FGM copula constructed with the representation in Theorem 3.2 with the vector of comonotonic rvs $\boldsymbol{I}^+$ is the EPD copula $C^{EPD} \in \mathcal{C}_d^{FGM}$, $d \geq 2$, that is,*

$$\boldsymbol{U} \preceq_{sm} \boldsymbol{U}^{EPD} \ \forall \ F_{\boldsymbol{U}} \in \mathcal{C}_d^{FGM}, \tag{20}$$

and

$$\boldsymbol{X} \preceq_{sm} \boldsymbol{X}^{EPD} \ \forall \ F_{\boldsymbol{X}} \in \mathcal{H}_d^{FGM}. \tag{21}$$

The expression of the EPD copula C^{EPD} is given by

$$C^{EPD}\left(\boldsymbol{u}\right) = \prod_{j=1}^{d} u_j \left(1 + \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} \sum_{1 \leq j_1 < \cdots < j_{2k} \leq d} \overline{u}_{j_1} \cdots \overline{u}_{j_{2k}}\right), \quad \boldsymbol{u} \in [0,1]^d, \tag{22}$$

where $\lfloor y \rfloor$ is the floor function returning the greatest integer smaller or equal to y. An alternative expression to (22) is

$$C^{EPD}\left(\boldsymbol{u}\right) = \frac{1}{2} \prod_{m=1}^{d} (1 - \overline{u}_m^2) + \frac{1}{2} \prod_{j=1}^{d} u_j^2, \quad \boldsymbol{u} \in [0,1]^d. \tag{23}$$

Proof. The relations in (20) and (21) follow from Theorem 4.2 and the result in (19). To derive C^{EPD}, we replace (18) in (11). The alternative expression of C^{EPD} in (23) results by using (11) with (18). □

Remark 4.4. One obtains the extremal positive dependence within bivariate FGM copulas with parameter $\theta_{12} = 1$. For C_d^{EPD} and $d > 2$, the parameters are such that the k-dependence parameters are 1 for k even and 0 for k odd. One can express the parameters compactly for both even and odd k-dependence parameters as $\theta_{j_1 \ldots j_k} = \left(1 + (-1)^k\right)/2$, for $1 \leq j_1 < \ldots j_k \leq d$ and $k = \{2, \ldots, d\}$.

5 Measures of multivariate association

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a d-vector of rvs with continuous marginals $H_1, ..., H_d$ and joint cdf $F_{\boldsymbol{X}}(\boldsymbol{x}) = C(H_1(x_1), \ldots, H_d(x_d))$, $\boldsymbol{x} \in \mathbb{R}^d$, where $C \in \mathcal{C}_d$. We define the d-variate independence copula by $C^{\perp}(\boldsymbol{u}) = \prod_{j=1}^{d} u_j$, $\boldsymbol{u} \in [0,1]^d$. Measures of multivariate association are closely related to notions of dependence, which are themselves parallel to multivariate dependence orders. The following are definitions of the three most common ones (see Definition 5.3.18 in Denuit et al. (2006)).

- $\boldsymbol{X}$ is positively lower orthant dependent (PLOD) if $C(\boldsymbol{u}) \geq C^{\perp}(\boldsymbol{u})$, for all $\boldsymbol{u} \in [0,1]^d$, that is, if the probability that the variables $\boldsymbol{X}$ simultaneously take small values is at least as great as it would be if they were independent.

- $\boldsymbol{X}$ is positively upper orthant dependent (PUOD) if $\overline{C}(\boldsymbol{u}) \geq \overline{C}^{\perp}(\boldsymbol{u})$, for all $\boldsymbol{u} \in [0,1]^d$, that is, if the probability that the variables $\boldsymbol{X}$ simultaneously take large values is at least as great as it would be if they were independent.

- $\boldsymbol{X}$ is positively orthant dependent (POD) if both inequalities hold.

When $d = 2$, Spearman's rho is defined by

$$\rho_S(X_1, X_2) = 12 \int_{[0,1]^2} C(u_1, u_2) \mathrm{d}C^{\perp}(u_1, u_2) - 3 = 12 \int_{[0,1]^2} C^{\perp}(u_1, u_2) \mathrm{d}C(u_1, u_2) - 3. \tag{24}$$

When $C \in \mathcal{C}_2^{FGM}$, we have $\rho_S(X_1, X_2) = \theta_{12}/3$, as mentioned in the introduction. In Nelsen (1996), the author analyzes two d-variate extensions of Spearman's rho defined in (24) in a multivariate setting :

$$\rho_d^{cL}(\boldsymbol{X}) = \frac{d+1}{2^d - (d+1)} \left[2^d \left(\int_{[0,1]^d} C(\boldsymbol{u}) \mathrm{d}C^{\perp}(\boldsymbol{u}) \right) - 1 \right] \tag{25}$$

and

$$\rho_d^{cU}(\boldsymbol{X}) = \frac{d+1}{2^d - (d+1)} \left[2^d \left(\int_{[0,1]^d} C^{\perp}(\boldsymbol{u}) \mathrm{d}C(\boldsymbol{u}) \right) - 1 \right]. \tag{26}$$

Hence, ρ_d^{cL} (ρ_d^{cU}) is an average of the positively lower (upper) orthant association.

We mention that (25) and (26) were initially introduced in Wolff (1980) and Joe (1990), respectively. The author of Nelsen (1996) introduced a third version of d-variate Spearman's rho that corresponds to the average of ρ_d^{cL} and ρ_d^{cU}, as follows:

$$\rho_d^{c}(\boldsymbol{X}) = \frac{\rho_d^{cL}(\boldsymbol{X}) + \rho_d^{cU}(\boldsymbol{X})}{2}. \tag{27}$$

As explained in Nelsen (1996), for $d = 2$, $\rho_2^{cL}(\boldsymbol{X}) = \rho_2^{cU}(\boldsymbol{X}) = \rho_2^{c}(\boldsymbol{X}) = \rho_S(\boldsymbol{X})$.

Following Schmid and Schmidt (2007) and Gijbels et al. (2021), we also define

$$\rho_d^{pw} = \frac{1}{\binom{d}{2}} \sum_{1 \leq j_1 < j_2 \leq d} \rho_S(X_{j_1}, X_{j_2})$$

as the average of bivariate Spearman's rhos.

For two recent reviews and analysis of ρ_d^{cL}, ρ_d^{cU}, and ρ_d^{pw}, we refer the reader to García-Gómez et al. (2021) and Gijbels et al. (2021), and the references therein. See also these two references for empirical applications of measures of multivariate associations. In Schmid and Schmidt (2007), the authors examine the estimation procedure of (25) and (26) based on the empirical copula, and they provide the empirical estimators of those two measures of multivariate associations. In Pérez and Prieto-Alaiz (2016), the authors propose and discuss alternatives to those estimators since they can take values out of the parameter space. The following result holds for d-variate Spearman's rhos.

Theorem 5.1. *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be such that $F_{\boldsymbol{X}}, F_{\boldsymbol{X}'} \in \mathcal{H}_d$. If $\boldsymbol{X} \preceq_{cL} \boldsymbol{X}'$, then $\rho_d^{cL}(\boldsymbol{X}) \leq \rho_d^{cL}(\boldsymbol{X}')$. If $\boldsymbol{X} \preceq_{cU} \boldsymbol{X}'$, then $\rho_d^{cU}(\boldsymbol{X}) \leq \rho_d^{cU}(\boldsymbol{X}')$. If $\boldsymbol{X} \preceq_{c} \boldsymbol{X}'$, then $\rho_d^{c}(\boldsymbol{X}) \leq \rho_d^{c}(\boldsymbol{X}')$ and $\rho_d^{pw}(\boldsymbol{X}) \leq \rho_d^{pw}(\boldsymbol{X}')$.*

Proof. The relationship $\rho_d^{cL}(\boldsymbol{X}) \leq \rho_d^{cL}(\boldsymbol{X}')$ follows both from the definition of ρ_d^{cL} and "$\preceq_c \Rightarrow \preceq_{cL}$". Also, $\rho_d^{cU}(\boldsymbol{X}) \leq \rho_d^{cU}(\boldsymbol{X}')$ follows from "$\preceq_c \Rightarrow \preceq_{cU}$" and Lemma 3.3.1 of Joe (1990). From the two previous inequalities and the definition of "$\preceq_c$", the inequality $\rho_d^{c}(\boldsymbol{X}) \leq \rho_d^{c}(\boldsymbol{X}')$ holds. Finally, since the concordance order is closed on marginalization and by the definition of ρ_d^{pw}, we obtain $\rho_d^{pw}(\boldsymbol{X}) \leq \rho_d^{pw}(\boldsymbol{X}')$. □

Theorem 5.1 holds because the measures studied in this section satisfy the axiom of concordance as defined in Schmid et al. (2010) and the axiom of ordering as defined in Gijbels et al. (2021), stating that that if $C_1 \preceq C_2$, then $\kappa(C_1) \leq \kappa(C_2)$, where κ is a multivariate association measure. Therefore, the results from this section follow directly from Theorem 4.2.

If $F_{\boldsymbol{X}} \in \mathcal{H}_d^{FGM}$ (that is, $C \in \mathcal{C}_d^{FGM}$), then (25) and (26) become

$$\rho_d^{cL}(\boldsymbol{X}) = \frac{d+1}{d^\star}\left[\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\left(\frac{1}{3}\right)^k\right] \tag{28}$$

and

$$\rho_d^{cU}(\boldsymbol{X}) = \frac{d+1}{d^\star}\left[\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\left(-\frac{1}{3}\right)^k\right], \tag{29}$$

where the denominator $2^d-(d+1)$ in both (28) and (29) coincides with the number of parameters $d^\star$ of a d-variate FGM copula as defined in (3). Expressions in (28) and (29) generalized those provided in Example 2 of Nelsen (1996) for $d=3$, and were also derived in the supplementary materials of Gijbels et al. (2021). Replacing (28) and (29) in (27), we find that

$$\rho_d(\boldsymbol{X}) = \frac{d+1}{d^\star}\left[\sum_{l=1}^{\lfloor\frac{d}{2}\rfloor}\sum_{1\leq j_1<\cdots<j_{2\times l}\leq d}\theta_{j_1\ldots j_{2\times l}}\left(\frac{1}{3}\right)^{2\times l}\right].$$

Note that both definitions in (28) and (29) use all the values of the dependence parameter vector $\boldsymbol{\theta}$. However, the k-dependence parameters, for $k\in\{3,5,7,\ldots\}$, does not contribute to the value of $\rho_d(\boldsymbol{X})$. One interpretation is that (28) and (29) aggregate some knowledge about the dependence structure of a d-variate FGM copula, either in the lower or the upper orthants.

For $d\in\{3,4,\ldots\}$, we decompose (28) and (29) as follows:

$$\rho_d^{cL}(\boldsymbol{X}) = \frac{d+1}{3d^\star}\sum_{1\leq j_1<j_2\leq d}\rho_S(X_{j_1},X_{j_2}) + \frac{d+1}{d^\star}\left[\sum_{k=3}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\left(\frac{1}{3}\right)^k\right] \tag{30}$$

and

$$\rho_d^{cU}(\boldsymbol{X}) = \frac{d+1}{3d^\star}\sum_{1\leq j_1<j_2\leq d}\rho_S(X_{j_1},X_{j_2}) + \frac{d+1}{d^\star}\left[\sum_{k=3}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}\theta_{j_1\ldots j_k}\left(-\frac{1}{3}\right)^k\right]. \tag{31}$$

Corollary 5.2. *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be such that $F_{\boldsymbol{X}}, F_{\boldsymbol{X}'} \in \mathcal{H}_d^{FGM}$. If $\boldsymbol{X} \preceq_{cL} \boldsymbol{X}'$, then $\rho_d^{cL}(\boldsymbol{X}) \leq \rho_d^{cL}(\boldsymbol{X}')$. If $\boldsymbol{X} \preceq_{cU} \boldsymbol{X}'$, then $\rho_d^{cU}(\boldsymbol{X}) \leq \rho_d^{cU}(\boldsymbol{X}')$. If $\boldsymbol{X} \preceq_{c} \boldsymbol{X}'$, then $\rho_d^{c}(\boldsymbol{X}) \leq \rho_d^{c}(\boldsymbol{X}')$ and $\rho_d^{pw}(\boldsymbol{X}) \leq \rho_d^{pw}(\boldsymbol{X}')$.*

Both (30) and (31) clearly exhibit that ρ_d^{cL} and ρ_d^{cU} aim to measure the global dependence between the components of $\boldsymbol{X}$. In other words, ρ_d^{cL} and ρ_d^{cU} capture the pairwise association through the Spearman's rhos for all pairs of $\boldsymbol{X}$ but also to measure the d-tuple-wise association, for $d\in\{3,4,\ldots\}$, among the components of $\boldsymbol{X}$. See Durante et al. (2014) for a discussion on the distinction between pairwise and global dependence.

Corollary 5.3. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{X}^{EPD}$ *be such that* $F_{\boldsymbol{X}}, F_{\boldsymbol{X}^{EPD}} \in \mathcal{H}_d$. *It follows that* $\rho_d^{cL}(\boldsymbol{X}) \leq \rho_d^{cL}(\boldsymbol{X}^{EPD})$ *and* $\rho_d^{cU}(\boldsymbol{X}) \leq \rho_d^{cU}(\boldsymbol{X}^{EPD})$, *where*

$$\rho_d^{cL}(\boldsymbol{X}^{EPD}) = \rho_d^{cU}(\boldsymbol{X}^{EPD}) = \rho_d^{c}(\boldsymbol{X}^{EPD}) = \frac{d+1}{d^\star}\left[\frac{1}{2}\left(\frac{2}{3}\right)^d + \frac{1}{2}\left(\frac{4}{3}\right)^d - 1\right]. \tag{32}$$

Moreover, $\rho_d^{pw}(\boldsymbol{X}) \leq \rho_d^{pw}(\boldsymbol{X}^{EPD}) = 1/3$.

Proof. The inequalities are a consequence of the combination of Theorem 4.3 and Theorem 5.1. It implies that $\rho_d^{cL}(\boldsymbol{X}^{EPD}) = \rho_d^{cU}(\boldsymbol{X}^{EPD}) = \rho_d^{c}(\boldsymbol{X}^{EPD})$. From (28), we find

$$\begin{aligned}
\rho_d^{cL}(\boldsymbol{X}) &= \frac{d+1}{d^\star}\left[\sum_{k=2}^{d}\binom{d}{k}\left(\frac{1}{3}\right)^k \frac{1}{2}\left(1+(-1)^k\right)\right] \\
&= \frac{d+1}{d^\star}\left[\frac{1}{2}\sum_{k=0}^{d}\binom{d}{k}\left(\frac{1}{3}\right)^k + \frac{1}{2}\sum_{k=0}^{d}\binom{d}{k}\left(-\frac{1}{3}\right)^k - 1\right] \\
&= \frac{d+1}{d^\star}\left[\frac{1}{2}\left(1+\frac{1}{3}\right)^d + \frac{1}{2}\left(1-\frac{1}{3}\right)^d - 1\right].
\end{aligned}$$

□

Remark 5.4. Even if $\rho_d^c(\boldsymbol{X}^+) = 1$ for any $d \in \{2, 3, \dots\}$, note from (32) that $\rho_d^c(\boldsymbol{X}^{EPD})$ is a strictly decreasing function of d for $d \in \{3, 4, \dots\}$. Indeed, the dominant term is $(4/3)^d$ in the numerator and 2^d in the denominator, so we have $\lim_{d\to\infty} \rho_d(\boldsymbol{X}^{EPD}) = 0$.

6 Bivariate FGM copulas

We aim to improve our understanding of the family of bivariate FGM copulas. To achieve this, the values of the joint pmf f_{I_1,I_2} of a pair (I_1, I_2) of Bernoulli rvs are gathered within a 4-dimensional vector $\mathbf{f}_{(I_1,I_2)}$ as follows:

$$\mathbf{f}_{(I_1,I_2)} = \left(f_{I_1,I_2}(0,0), f_{I_1,I_2}(0,1), f_{I_1,I_2}(1,0), f_{I_1,I_2}(1,1)\right). \tag{33}$$

Given the constraints $f_{I_1,I_2}(i_1,i_2) \geq 0$, $(i_1,i_2) \in \{0,1\}^2$, and $\sum_{(i_1,i_2)\in\{0,1\}^2} f_{I_1,I_2}(i_1,i_2) = 1$, the pmf of a bivariate Bernoulli rvs has three free parameters. We represent the set of all admissible values of $(f_{I_1,I_2}(0,0), f_{I_1,I_2}(0,1), f_{I_1,I_2}(1,0))$ as the tetrahedron (inspired from Figure 1 of Diaconis (1977)) depicted within the 3-dimensional cartesian graph in Figure 1. The tetrahedron represents a convex hull of four extremal points that correspond to the four vertices (see Rockafellar (2015) for details on convex analysis). Hence, the four vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ of the tetrahedron in Figure 1 are the extremal points of the set of all admissible values of the 4-dimensional vector $\mathbf{f}_{(I_1,I_2)}$ given by $(0,0,0,1)$, $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,1,0)$, respectively.

In the specific case when $F_{(I_1,I_2)} \in \mathcal{B}_2$, the following additional constraints hold:

$$\sum_{i_1\in\{0,1\}} f_{I_1,I_2}(i_1,0) = \sum_{i_1\in\{0,1\}} f_{I_1,I_2}(i_1,1) = \sum_{i_2\in\{0,1\}} f_{I_1,I_2}(0,i_2) = \sum_{i_2\in\{0,1\}} f_{I_1,I_2}(1,i_2) = \frac{1}{2}.$$

This means that symmetric bivariate Bernoulli distributions have only one free parameter, say $f_{I_1,I_2}(0,0) \in [0, 1/2]$.

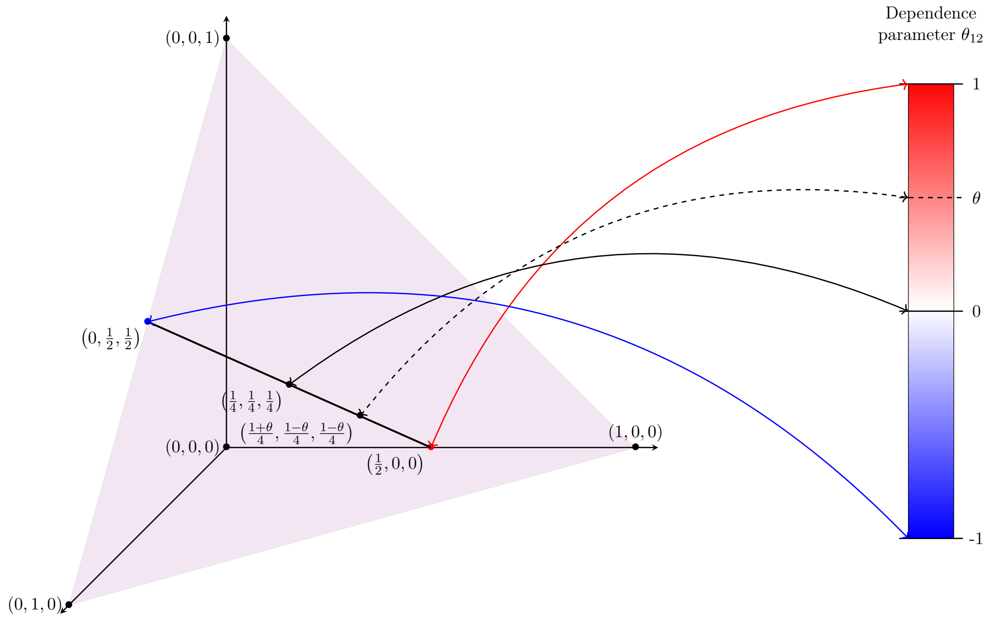


Figure 1: Tetrahedron: set of all possible values of $f_{(I_1,I_2)}(i_1,i_2)$, $(i_1,i_2)\in\{0,1\}^2$, where a point is a 4-dimensional vector. Segment between $(1/2,0,0,1/2)$ and $(0,1/2,1/2,0)$: Fréchet's class $\mathcal{B}_2$.

Consider the lower and upper bounds of the Fréchet class $\mathcal{B}_2$ with cdfs $F_{(I_1^-,I_2^-)}(i_1,i_2) = \max\left(F_{I_1}(i_1)+F_{I_2}(i_2)-1;0\right)$ and $F_{(I_1^+,I_2^+)}(i_1,i_2)=\min\left(F_{I_1}(i_1),F_{I_2}(i_2)\right)$. In Figure 1, the points $(1/2,0,0)$ and $(0,1/2,1/2)$ correspond to the values of $\mathbf{f}_{(I_1^+,I_2^+)}$ and $\mathbf{f}_{(I_1^-,I_2^-)}$ given by $(1/2,0,0,1/2)$ and $(0,1/2,1/2,0)$, respectively. It follows that the segment between the points $(1/2,0,0)$ and $(0,1/2,1/2)$ corresponds to Fréchet's class $\mathcal{B}_2$.

We define the pair of continuous positive rvs (Y_1,Y_2) as in (6) with

$$Y_1 = Z_{1,0}+I_1Z_{1,1} \quad \text{and} \quad Y_2 = Z_{2,0}+I_2Z_{2,1}.$$

By Theorem 3.1, (Y_1,Y_2) follows a bivariate exponential distribution with $Y_j\sim Exp\,(1)$, $j=1,2$, and joint cdf $F_{(Y_1,Y_2)}$ given by

$$\sum_{(i_1,i_2)\in\{0,1\}^2} f_{I_1,I_2}(i_1,i_2)\left(1-e^{-2x_1}\right)^{1-i_1}\left(1-e^{-2x_2}\right)^{1-i_2}\left(1-e^{-x_1}\right)^{2i_1}\left(1-e^{-x_2}\right)^{2i_2}, \tag{34}$$

for $(x_1,x_2)\in\mathbb{R}_+^2$. The next result is adapted from Theorem 3.2.

Corollary 6.1. *Using Theorem 3.2, we establish that $F_{(Y_1,Y_2)}$ as given in (34) is also*

$$F_{(Y_1,Y_2)}\left(x_1,x_2\right)=\left(1-e^{-x_1}\right)\left(1-e^{-x_2}\right)+\theta_{12}\left(1-e^{-x_1}\right)\left(1-e^{-x_2}\right)e^{-x_1}e^{-x_2},$$

for $(x_1,x_2)\in\mathbb{R}_+^2$, with

$$\theta_{12}=4f_{I_1,I_2}(0,0)-1, \tag{35}$$

that implies $F_{(Y_1,Y_2)}\in\mathcal{E}_2^{FGM}$. Furthermore, it follows that the corresponding copula $C\in\mathcal{C}_2^{FGM}$ where C is given by

$$C\left(u_1,u_2\right)=u_1u_2+\theta_{12}u_1u_2\overline{u}_1\overline{u}_2,\quad (u_1,u_2)\in[0,1]^2. \tag{36}$$

Hence, as an interpretation of Corollary 6.1, fixing values of the probabilities of the vector $\mathbf{f}_{(I_1,I_2)}$ in (33) leads through (35) to the value of the dependence parameter θ_{12} of a FGM copula C given in (36).

On the contrary, one may also ask how to get the values of the probabilities in $\mathbf{f}_{(I_1,I_2)}$ given a dependence parameter $\theta_{12} \in [-1, 1]$? The answer also results from Theorem 3.2.

Corollary 6.2. *If $C \in \mathcal{C}_2^{FGM}$ with a dependence parameter $\theta_{12} \in [-1, 1]$, then we write the entries of the vector $\mathbf{f}_{(I_1,I_2)}$ in (33) explicitly in terms of θ_{12} as follows:*

$$\mathbf{f}_{(I_1,I_2)} = \left(\frac{1+\theta_{12}}{4}, \frac{1-\theta_{12}}{4}, \frac{1-\theta_{12}}{4}, \frac{1+\theta_{12}}{4} \right).$$

Proof. Given the constraints on the values of the probabilities in $\mathbf{f}_{\boldsymbol{I}}$, (33) becomes

$$\mathbf{f}_{(I_1,I_2)} = \left(f_{I_1,I_2}(0,0), \frac{1}{2} - f_{I_1,I_2}(0,0), \frac{1}{2} - f_{I_1,I_2}(0,0), f_{I_1,I_2}(0,0) \right), \tag{37}$$

where $f_{I_1,I_2}(0,0) \in \left[0, \frac{1}{2}\right]$. Combining (35) and (37), we have

$$\begin{aligned} \theta_{12} &= f_{I_1,I_2}(0,0) - f_{I_1,I_2}(0,1) - f_{I_1,I_2}(1,0) + f_{I_1,I_2}(1,1) \\ &= 2f_{I_1,I_2}(0,0) - 2\left(\frac{1}{2} - f_{I_1,I_2}(0,0)\right). \end{aligned} \tag{38}$$

From (38), we find that $f_{I_1,I_2}(0,0) = (1+\theta_{12})/4$. Replacing the latter in (37), we obtain the desired result. □

Given Corollary 6.2, we have that each value of $\theta_{12} \in [-1, 1]$ uniquely defines an element of $\mathcal{B}_2$ and a member of the family of bivariate FGM copulas. In summary, Corollaries 6.1 and 6.2 define a one-to-one correspondence between the value of the dependence parameter θ_{12} and the four values of the joint pmf of the pair of rvs (I_1, I_2) (or equivalently, its distribution). That bijection is depicted in Figure 1.

In addition, since the copula is linear in θ_{12}, one can represent any bivariate copula as a convex combination of the two extremal copulas constructed with $\mathbf{f}_{(I_1^+,I_2^+)}$ and $\mathbf{f}_{(I_1^-,I_2^-)}$, leading to the parameters $\theta_{12} = 1$ and $\theta_{12} = -1$, respectively.

7 Trivariate FGM copulas

While the set of admissible parameters of the bivariate FGM copula is simple ($\theta_{12} \in [-1, 1]$), the condition $\boldsymbol{\theta} \in \mathcal{T}_3$ isn't trivial to conceptualize, that is, it isn't intuitively obvious if a set of parameters $(\theta_{12}, \theta_{13}, \theta_{23}, \theta_{123})$ will satisfy the eight inequalities in $\mathcal{T}_3$. In this section, we study more deeply the family of trivariate FGM copulas by illustrating the results of Theorem 3.2 in the trivariate case. When $d = 3$ in (5), a FGM copula C is given by

$$C(\boldsymbol{u}) = u_1u_2u_3 + u_1u_2u_3\left(\theta_{12}\overline{u}_1\overline{u}_2 + \theta_{13}\overline{u}_1\overline{u}_3 + \theta_{23}\overline{u}_2\overline{u}_3 + \theta_{123}\overline{u}_1\overline{u}_2\overline{u}_3\right), \quad \boldsymbol{u} \in [0,1]^3, \tag{39}$$

where (8) becomes

$$\theta_{j_1j_2} = 4f_{I_{j_1},I_{j_2}}(0,0) - 1, \quad 1 \le j_1 < j_2 \le 3 \tag{40}$$

$$\theta_{123} = 8f_{I_1,I_2,I_3}(0,0,0) - \theta_{12} - \theta_{13} - \theta_{23} - 1 \tag{41}$$

$$= -8f_{I_1,I_2,I_3}(1,1,1) + \theta_{12} + \theta_{13} + \theta_{23} + 1. \tag{42}$$

In the remainder of this section, we study specific dependence structures of the trivariate FGM copula.

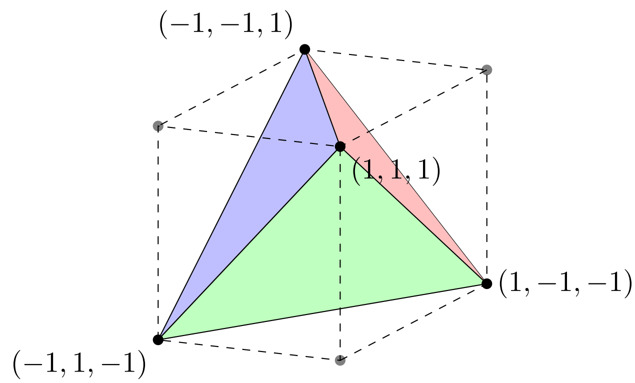


Figure 2: Admissible parameters for the trivariate subfamily with non-null 2-dependence and null 3-dependence parameters, that is, $\mathcal{T}_{3,2}$.

7.1 Subfamily with non-null 2-dependence and null 3-dependence parameters

In this subsection, we consider trivariate FGM copulas with only 2-dependence parameters, that is, $\theta_{123} = 0$. The structure within this subfamily does not mean that the vector (U_1, U_2, U_3) is independent, but that only 2-dependent parameters $\theta_{j_1 j_2}$, $1 \leq j_1 < j_2 \leq 3$ have non-zero values. Note that $\rho_3^{cL}(\boldsymbol{U}) = \rho_3^{cU}(\boldsymbol{U}) = \rho_3^{c}(\boldsymbol{U}) = \rho_3^{pw}(\boldsymbol{U})$ for this subfamily. We name $\mathcal{T}_{3,2}$ the set of valid non-null 2-dependence and null 3-dependence parameters, with

$$\mathcal{T}_{3,2} = \left\{ (\theta_{12}, \theta_{13}, \theta_{23}) \in \mathbb{R}^3 : 1 + \sum_{1 \leq j_1 < j_2 \leq 3} \theta_{j_1 j_2} \varepsilon_{j_1} \varepsilon_{j_2} \geq 0 \right\},$$

for $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \in \{-1, 1\}^3$. Note that $\mathcal{T}_{3,2} \subset \mathcal{T}_3$. The constraints in (4) implies that $(\theta_{12}, \theta_{13}, \theta_{23}) \subseteq [-1, 1]^3$, which corresponds to the cube in dotted lines from Figure 2. However, for the specific set of inequalities $\mathcal{T}_{3,2}$ forms a tetrahedron with vertices $(\theta_{12}, \theta_{13}, \theta_{23}) = (1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$. We present this tetrahedron in Figure 2. The great advantage of the tetrahedron is that one may interpret the admissible parameters $\mathcal{T}_{3,2}$ geometrically as a 3-simplex, instead of a set of four inequalities.

Note that C_3^{EDP} defined in Theorem 4.3 is an element of the subfamily of trivariate FGM copulas studied in this subsection, with $\theta_{12} = \theta_{13} = \theta_{23} = 1$ and $\theta_{123} = 0$. The copula C^{EPD} corresponds to the vertex $(1, 1, 1)$ of the tetrahedron in Figure 2.

Consider the vertex $(1, -1, -1)$. Applying Theorem 3.2, the associated pmf is

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \begin{cases} \frac{1}{2}, & \boldsymbol{i} \in \{(0,0,1), (1,1,0)\} \\ 0, & \text{otherwise} \end{cases}$$

and observe that the pairs (I_1, I_2) and (I_1, I_3) are countercomonotonic rvs and the pair (I_2, I_3) is comonotonic. Written differently, the vertex $(1, -1, -1)$ has a one-to-one association with the cdf

$$F_{\boldsymbol{I}}(\boldsymbol{i}) = \max\left(\min\left[F_{I_2}(i_2), F_{I_3}(i_3)\right] + F_{I_1}(i_1) - 1; 0\right), \quad \boldsymbol{i} \in \{0, 1\}^3.$$

In addition, we have $\rho_3^{cL}(\boldsymbol{X}) = \rho_3^{cU}(\boldsymbol{X}) = \rho_3^{c}(\boldsymbol{X}) = \rho_3^{pw}(\boldsymbol{X}) = -1/9$. One obtains identical results for the vertices $(-1, 1, -1)$ and $(-1, -1, 1)$ of Figure 2 by appropriately interchanging the components of (I_1, I_2, I_3).

7.2 Subfamily with 2-independence

Let $\boldsymbol{U} = (U_1, U_2, U_3)$ be a vector of rvs with $F_{\boldsymbol{U}} = C$ and $U_j \sim Unif\,(0,1)$, $j \in \{1,2,3\}$. In this section, we consider the subfamily of dependence parameters from $\mathcal{T}_3$ with $\theta_{12} = \theta_{13} = \theta_{23} = 0$ and $\theta_{123} \in [-1,1]$. In other words, trivariate FGM copulas with 2-independence have 2-dependence parameters equal to 0 and a free 3-dependence parameter on the segment $[-1,1]$. We note that $\rho_S(U_{j_1}, U_{j_2}) = 0$ for $1 \leq j_1 < j_2 \leq 3$. Denote $A^{(1)}$, $A^{(-1)} \subset \{0,1\}^3$ such that

$$A^{(1)} = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$$
$$A^{(-1)} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\},$$

that is, the sum of the elements are even for vectors in $A^{(1)}$ and odd for vectors in $A^{(-1)}$. Let $f_{\boldsymbol{I}^{(1)}}$ be defined such that $f_{\boldsymbol{I}^{(1)}}(i_1,i_2,i_3) = 1/4$, for $(i_1,i_2,i_3) \in A^{(1)}$, and $f_{\boldsymbol{I}^{(1)}}(i_1,i_2,i_3) = 0$, for $(i_1,i_2,i_3) \in A^{(-1)}$. Similarly, let $f_{\boldsymbol{I}^{(-1)}}$ be defined such that $f_{\boldsymbol{I}^{(-1)}}(i_1,i_2,i_3) = 1/4$, for $(i_1,i_2,i_3) \in A^{(-1)}$ and $f_{\boldsymbol{I}^{(-1)}}(i_1,i_2,i_3) = 0$, for $(i_1,i_2,i_3) \in A^{(1)}$. Then, by combining (39) and (41), it follows that the two resulting trivariate FGM copulas are given by

$$C^{(1)}(\boldsymbol{u}) = u_1u_2u_3 + u_1u_2u_3\overline{u}_1\overline{u}_2\overline{u}_3;$$
$$C^{(-1)}(\boldsymbol{u}) = u_1u_2u_3 - u_1u_2u_3\overline{u}_1\overline{u}_2\overline{u}_3, \quad \boldsymbol{u} \in [0,1]^3.$$

Now, we define $\mathbf{f}_{\boldsymbol{I}^{(\theta_{123})}}$ such that

$$f_{\boldsymbol{I}^{(\theta_{123})}}(i_1,i_2,i_3) = \frac{1+\theta_{123}}{2} f_{\boldsymbol{I}^{(1)}}(i_1,i_2,i_3) + \frac{1-\theta_{123}}{2} f_{\boldsymbol{I}^{(-1)}}(i_1,i_2,i_3), \quad (i_1,i_2,i_3) \in \{0,1\}^3, \tag{43}$$

with $\theta_{123} \in [-1,1]$. Inserting (43) in (39), (40) and (41), the resulting copula is

$$C^{(\theta_{123})}(\boldsymbol{u}) = u_1u_2u_3 + \theta_{123}u_1u_2u_3\overline{u}_1\overline{u}_2\overline{u}_3, \quad \boldsymbol{u} \in [0,1]^3. \tag{44}$$

From (44), we observe that (U_1,U_2), (U_1,U_3), (U_2,U_3) are pairs of independent rvs. However, the components of $\boldsymbol{U}$ are dependent. Also, (44) corresponds to the copula described at page 554 of Cambanis (1977). See also Section 1.6.2 of Rüschendorf (2013) for FGM copulas that are $(d-1)$-independent but not d-independent.

Corollary 7.1. *Let $-1 \leq \theta_{123} < \theta'_{123} \leq 1$, then $\boldsymbol{I}^{(\theta_{123})} \preceq_{cL} \boldsymbol{I}^{(\theta'_{123})}$ and $\boldsymbol{U}^{(\theta_{123})} \preceq_{cL} \boldsymbol{U}^{(\theta'_{123})}$, and $\boldsymbol{I}^{(\theta'_{123})} \preceq_{cU} \boldsymbol{I}^{(\theta_{123})}$ and $\boldsymbol{U}^{(\theta'_{123})} \preceq_{cU} \boldsymbol{U}^{(\theta_{123})}$. Moreover, $\boldsymbol{I}^{(\theta_{123})} \preceq_{sm} \boldsymbol{I}^{(\theta'_{123})}$ and $\boldsymbol{U}^{(\theta_{123})} \preceq_{sm} \boldsymbol{U}^{(\theta'_{123})}$ do not hold.*

Proof. See Example 3.8.4 in Müller and Stoyan (2002). One has $F_{\boldsymbol{I}^{(1)}}(\boldsymbol{i}) \geq F_{\boldsymbol{I}^{(-1)}}(\boldsymbol{i})$ and $\overline{F}_{\boldsymbol{I}^{(1)}}(\boldsymbol{i}) \leq \overline{F}_{\boldsymbol{I}^{(-1)}}(\boldsymbol{i})$ for all $\boldsymbol{i} \in \{0,1\}^3$, where $\overline{F}_{\boldsymbol{I}}(\boldsymbol{i})$ is the survival function of $\boldsymbol{I}$. □

Corollary 7.2. *If $-1 \leq \theta_{123} < \theta'_{123} \leq 1$, then*

1. $\rho_d^{cL}\left(\boldsymbol{U}^{(\theta_{123})}\right) \leq \rho_d^{cL}\left(\boldsymbol{U}^{(\theta'_{123})}\right)$;

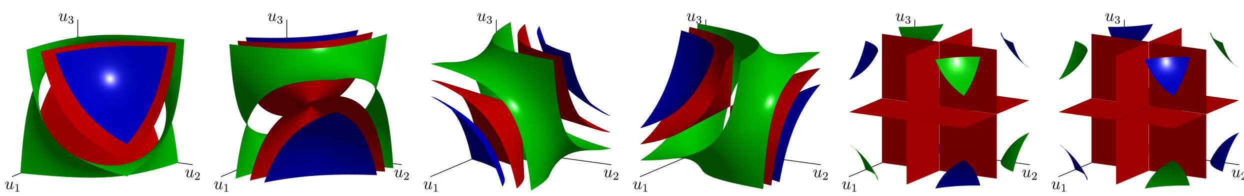
2. $\rho_d^{cU}\left(\boldsymbol{U}^{(\theta_{123})}\right) \geq \rho_d^{cU}\left(\boldsymbol{U}^{(\theta'_{123})}\right)$.

Proof. Applying Lemma 5.1 with Corollary 7.1 yields the desired result. Alternatively, one computes $\rho_3^{cL}(\boldsymbol{U}) = \theta_{123}/27$, which is an increasing function of θ_{123}, and $\rho_3^{cU}(\boldsymbol{U}) = -\theta_{123}/27$, which is a decreasing function of θ_{123}. □

Observe that $\rho_3^{c}(\boldsymbol{U}) = \rho_3^{pw}(\boldsymbol{U}) = 0$. Also note that the authors of Schmid and Schmidt (2007) write $\rho_d^{cU}(\boldsymbol{X}) = \theta_{123}/6^3$, which is a misprint since it is 2^3 times too small.

$\boldsymbol{\theta}^{extr}$	θ_{12}	θ_{13}	θ_{23}	θ_{123}	$f_{0,0,0}$	$f_{1,0,0}$	$f_{0,1,0}$	$f_{0,0,1}$	$f_{1,1,0}$	$f_{1,0,1}$	$f_{0,1,1}$	$f_{1,1,1}$
a	1	1	1	0	1/2	–	–	–	–	–	–	1/2
b	1	-1	-1	0	–	–	–	1/2	1/2	–	–	–
c	-1	-1	1	0	–	1/2	–	–	–	–	1/2	–
d	-1	1	-1	0	–	–	1/2	–	–	1/2	–	–
e	0	0	0	-1	1/4	–	–	–	1/4	1/4	1/4	–
f	0	0	0	1	–	1/4	1/4	1/4	–	–	–	1/4

Table 1: Set of six extremal parameter vectors for trivariate FGM copulas.



(a) (b) (c) (d) (e) (f)

Figure 3: Three dimensional contour surfaces of the extremal points $\boldsymbol{\theta}^{extr}$ for trivariate FGM copulas.

7.3 Extremal parameters for trivariate FGM copulas

In Section 6, we mention that any bivariate FGM copula is a convex combination of two bivariate FGM copulas since the parameters θ_{12} lie between two extremal points -1 and 1. Similarly, one can represent any trivariate FGM copula $C \in C_3^{FGM}$ as a convex combination of six extremal FGM copulas. In Table 1, we provide the six dependence parameter vectors $\boldsymbol{\theta}$ and the corresponding values of $f_{\boldsymbol{I}}$ by applying (9) of Theorem 8. Row (a) represents the extremal positive dependent element detailed in Section 4.2. Rows (b), (c) and (d) represent extreme negative dependent elements, while rows (e) and (f) represent the two extreme parameter vectors for the subfamily with 2-independence.

To interpret the shape of the trivariate copula density function, we introduce 3-dimensional contour surfaces in Figure 3. Let $\boldsymbol{u}_\alpha$ be the set of points (u_1, u_2, u_3) such that the copula density function $c(u_1, u_2, u_3) = \alpha$, for $\alpha \geq 0$. Then, the surface of the points $\boldsymbol{u}_\alpha$ forms the contour surface at level α. The contour surface of Figure 3 (color online) presents the contours of $\boldsymbol{u}_{0.5}$ in blue, $\boldsymbol{u}_1$ in red and $\boldsymbol{u}_{1.5}$ in green. Then, $3c(\boldsymbol{u}_{0.5}) = c(\boldsymbol{u}_{1.5})$, meaning a point is three times as likely to fall on a green surface than on a blue surface.

The geometric shape of the 3-dimensional contour surfaces for extremal points of 2-dependence zero 3-dependence (extremal points (a) to (d)) are identical, but rotated along the u_3 axis by 90 degrees. The plot (a) represents the extreme positive dependence of the trivariate FGM copula, and the orientation of the contours have blue surfaces closest to the (0, 0, 0) and (1, 1, 1) corners. Similarly, plot (b) has blue contour surfaces closest to the (0, 0, 1) and (1, 1, 0) corners. For the subfamily with 2-independence (plots (e) and (f)), the copulas have a constant value of 1 along the three axis.

8 Subfamily of FGM copulas: the Markov-Bernoulli process

In this section we aim to understand the dependence structure within a specific subfamily of FGM copulas induced through a subfamily of multivariate Bernoulli distributions. With the help of the one-to-one correspondence between the class $\mathcal{B}_d$ and $\mathcal{C}_d^{FGM}$, we are able to reveal the dependence structures that are concealed in the values of $\boldsymbol{\theta} \in \mathcal{T}_d$. Examples of multivariate Bernoulli distributions can be found in Chapters 7 and 8 of Joe (1997).

This section introduces a subfamily of single-parameter FGM copulas with serial dependence, constructed with a multivariate Bernoulli model that also exhibits serial dependence. Let $\{I_j, j \in \mathbb{N}_+\}$ form a stationary Markov chain with a state space $\{0,1\}$, also called Markov chain of order 1 for binary time series in (Joe, 1997, pp. 246-248), with transition probability matrix

$$\mathbb{P} = \begin{pmatrix} p_{0|0} & p_{0|1} \\ p_{1|0} & p_{1|1} \end{pmatrix} = \begin{pmatrix} 1-(1-\alpha)\,q & (1-\alpha)\,q \\ (1-\alpha)\,(1-q) & \alpha+(1-\alpha)\,q \end{pmatrix},$$

dependence parameter $\alpha \in [-1,1]$, and where $p_{i_j|i_{j-1}} = \Pr(I_j = i_j | I_{j-1} = i_{j-1})$, $j \in \{2,3,\dots\}$. We call this model the Markov-Bernoulli model for the remainder of this paper. The initial probability associated to $\mathbb{P}$ is $\Pr(I_1 = 0) = 1/2$. If $\alpha = 0$, then $\{I_j, j \in \mathbb{N}_+\}$ is a sequence of iid rvs. For $\alpha = 1$, $\{I_j, j \in \mathbb{N}_+\}$ forms a sequence of comonotononic rvs.

In Cossette et al. (2003), the authors show

$$\mathbb{P}^{[h]} = \begin{pmatrix} p^{[h]}_{0|0} & p^{[h]}_{1|0} \\ p^{[h]}_{0|1} & p^{[h]}_{1|1} \end{pmatrix} = \begin{pmatrix} \frac{1+\alpha^h}{2} & \frac{1-\alpha^h}{2} \\ \frac{1-\alpha^h}{2} & \frac{1+\alpha^h}{2} \end{pmatrix},$$

where $p^{[h]}_{i_j|i_{j-1}} = \Pr(I_j = i_j | I_{j-1} = i_{j-1})$, $j \in \{2,3,\dots\}$ and $h \in \mathbb{N}_+$. The covariance between I_j and I_{j+h} is $Cov\,(I_j, I_{j+h}) = q(1-q)\alpha^h$, for $j \in \{2,3,\dots\}$ and $h \in \mathbb{N}_+$. The pmf of the vector $\boldsymbol{I}$ from a Markov-Bernoulli process is

$$f_{\boldsymbol{I}}\,(\boldsymbol{i}) = \frac{1}{2}\prod_{m=2}^{d} p_{i_m|i_{m-1}}, \quad i \in \{0,1\}^d. \tag{45}$$

The following proposition presents the expression of a FGM copula for the Markov-Bernoulli model and is proved in Section 11.

Proposition 8.1. *The expression of a d-variate FGM copula constructed with the d-dimensional vector $\boldsymbol{I}$ from a Markov-Bernoulli process with pmf in (45) is*

$$C\,(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(1 + \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} \sum_{1 \le j_1 < \cdots < j_{2k} \le d} \alpha^{\gamma_{j_1 \cdots j_{2k}}} \overline{u}_{j_1} \dots \overline{u}_{j_{2k}}\right), \quad \boldsymbol{u} \in [0,1]^d,$$

where $\gamma_{j_1 \dots j_{2k}} = \sum_{l=1}^{k} (j_{2l} - j_{2l-1})$. The k-dependence parameters are 0 for k odd and

$$\theta_{j_1 \dots j_{2k}} = \alpha^{\gamma_{j_1 \cdots j_{2k}}}, \tag{46}$$

for $1 \le j_1 < \cdots < j_{2k} \le d$ and $k \in \{1, \dots, \lfloor d/2 \rfloor\}$. Alternatively, we have

$$C\,(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(\sum_{\boldsymbol{i} \in \{0,1\}^d} \frac{1}{2^d} \prod_{j=2}^{d} \left(1 + (-1)^{i_j - i_{j-1}}\alpha\right) \prod_{l=1}^{d} \left(1 + (-1)^{i_l}\overline{u}_l\right)\right), \quad \boldsymbol{u} \in [0,1]^d. \tag{47}$$

When $d = 3$, we have the following special cases. For $\alpha = 1$, it follows that $\boldsymbol{I}$ is a vector of comonotonic rvs and we have $\theta_{12} = \theta_{13} = \theta_{23} = 1$ and $\theta_{123} = 0$, corresponding to the extremal positive dependence and row (a) of Table 1. For $\alpha = -1$, we have $\theta_{12} = \theta_{23} = -1$, $\theta_{13} = 1$

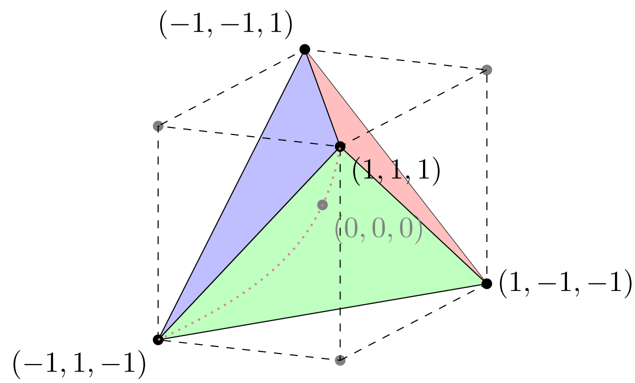


Figure 4: Segment of admissible parameters within the trivariate Markov-Bernoulli FGM copula.

and $\theta_{123} = 0$, corresponding to row (d) of Table 1. Since $\theta_{123} = 0$, trivariate Markov-Bernoulli FGM copulas belong to the subfamily studied in Section 7.1. In Figure 4, one finds a modified version of Figure 2, showing the segment of admissible dependence parameters for the subfamily of trivariate Markov-Bernoulli FGM copulas. The segment forms a quadratic curve connecting the vertex $(-1, 1, -1)$ for $\alpha = -1$, passing through quadratic function's vertex at $(0, 0, 0)$ for $\alpha = 0$, finishing at the vertex $(1, 1, 1)$ for $\alpha = 1$.

The following corollary establishes dependence order properties within the subfamily of Markov-Bernoulli distributions and Markov-Bernoulli FGM copulas.

Corollary 8.2. *Let $\boldsymbol{I}$ and $\boldsymbol{I}'$ be two Markov-Bernoulli d-dimensional vectors with pmf defined in (45) with parameters α and α' respectively. If $-1 \leq \alpha < \alpha' < 1$, then $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$ and $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$.*

Proof. To show $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$, we adapt a proof of a similar result presented in Cossette et al. (2020). Let

$$\mathbb{P}' = \begin{pmatrix} 1 - (1 - \alpha')\, q & (1 - \alpha')\, q \\ (1 - \alpha')\,(1 - q) & \alpha' + (1 - \alpha')\, q \end{pmatrix}$$

and $\boldsymbol{I}$ be a 2×2 identity matrix. Fix $c = (1 - \alpha')/(1 - \alpha)$. Since $-1 \leq \alpha < \alpha' < 1$, it implies that $c \in (0, 1)$. Because $\mathbb{P}'$ is of the form $\mathbb{P}' = (1 - c)\,\boldsymbol{I} + c\mathbb{P}$ and since I_1 (and I_2) is stochastically increasing in I_2 (in I_1), the first result follows from Corollary 3.1 of Hu and Pan (2000). The relation $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$ follows from $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$ and Theorem 4.2. □

Remark 8.3. For a copula $C \in C_d^{FGM}$, dependence ordering depends on all $d^\star$ dependence parameters simultaneously, thus it may be difficult to order FGM copulas in general. By limiting oneself to the subclass of Markov-Bernoulli FGM copulas, along with the result from Corollary 8.2, one can establish $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$ directly with a single parameter $-1 \leq \alpha < \alpha' \leq 1$, even when d is large and the dependence parameter vector $\boldsymbol{\theta}$ is very large. One can say that the subfamily is increasing in the sense of supermodular ordering with respect to α, for $-1 \leq \alpha \leq 1$.

Example. Let $(I_1, I_2, I_3, I_4, I_5, I_6)$ form a vector of rvs constructed from the Markov-Bernoulli model with pmf as in (45). Let $\boldsymbol{X}$ be a vector of rvs with cdf $F_{\boldsymbol{X}} \in \mathcal{H}_6^{FGM}$. Applying (46), the dependence parameter vector of $\boldsymbol{X}$ under the natural formulation is

$$\begin{aligned}
\theta_{j_1 j_2} &= \alpha^{j_2 - j_1}, \quad 1 \leq j_1 < j_2 \leq 6; \\
\theta_{j_1 j_2 j_3} &= 0, \quad 1 \leq j_1 < j_2 < j_3 \leq 6; \\
\theta_{j_1 j_2 j_3 j_4} &= \alpha^{j_2 - j_1 + j_4 - j_3} = \theta_{j_1 j_2} \theta_{j_3 j_4}, \quad 1 \leq j_1 < j_2 < j_3 < j_4 \leq 6; \\
\theta_{j_1 j_2 j_3 j_4 j_5} &= 0, \quad 1 \leq j_1 < \cdots < j_5 \leq 6; \\
\theta_{123456} &= \alpha^3.
\end{aligned}$$

Note that $\theta_{j_1 j_2} \in \{\alpha, \ldots, \alpha^5\}$, for $1 \leq j_1 < j_2 \leq 6$, and $\theta_{j_1 j_2 j_3 j_4} \in \{\alpha^2, \alpha^3, \alpha^4\}$, for $1 \leq j_1 < j_2 < j_3 < j_4 \leq 6$. Inserting the dependence parameter vector into (28) along with elementary algebra, one has

$$\rho_6^c(\boldsymbol{X}) = \rho_6^{cL}(\boldsymbol{X}) = \rho_6^{cU}(\boldsymbol{X}) = \frac{7}{57}\left[\frac{5\alpha + 4\alpha^2 + 3\alpha^3 + 2\alpha^4 + \alpha^5}{9} + \frac{6\alpha^2 + 6\alpha^3 + 3\alpha^4}{9^2} + \frac{\alpha^3}{9^3}\right],$$

which leads to the minimal value $\rho_6^c(\boldsymbol{X}) = -2000/54711$ for $\alpha = -1$ and the maximal value $\rho_6^c(\boldsymbol{X}) = \rho_6^c(\boldsymbol{X}^{EDP}) = 6413/28178$ for $\alpha = 1$.

9 Sampling

Sampling of vectors of large-dimensional random variables is essential in the context of numerous applications, especially in actuarial science, quantitative risk management, hydrology, etc. In Subsection 9.1, we briefly recall the conditional approach, with the contribution of Ota and Kimura (2021). In Subsection 9.2, we propose a new approach based on the stochastic representation derived in Theorem 3.2. In Subsection 9.3, we conclude by comparing the performance of the two algorithms in a numerical illustration.

9.1 Conditional sampling method

To our knowledge, the general method to simulate samples of $\boldsymbol{U} = (U_1, \ldots, U_d)$ when $F_{\boldsymbol{U}} = C \in \mathcal{C}_d^{FGM}$ is the so-called *conditional sampling method* (see Algorithm 5.1.2 in Mai and Scherer (2014)). The conditional sampling method is based on the multivariate distributional transform, also called Rosenblatt transform, introduced by Rosenblatt (1952) for absolutely continuous distributions. Later, the author of Ruschendorf (1981) generalized this transform to general distributions. Details on multivariate distributional transforms and its applications for simulations can be found in Section 1.3 of Rüschendorf (2013). Define the conditional distribution function as

$$C_{j|1,\ldots,j-1}(u_j|u_1, \ldots, u_{j-1}) = \Pr(U_j \leq u_j | U_1 = u_1, \ldots, U_{j-1} = u_{j-1}), \quad j \in \{2, \ldots, d\}$$

and $u_j \in [0, 1]$. The conditional sampling method consists of recursively computing

$$u_j = C^{-1}_{j|1,\ldots,j-1}(v_d|u_1, \ldots, u_{j-1}), \quad j \in \{2, \ldots, d\}, \tag{48}$$

where $u_1, v_2, \ldots, v_d$ are samples from a standard uniform distribution. Then, the vector $(u_1, \ldots, u_d)$ is a sample from a d-variate FGM distribution. In Section 5 of Ota and Kimura (2021), the authors show that (48) has a quadratic form, and present a method to compute the inverse conditional distribution function using the quadratic formula

$$u_j = C^{-1}_{j|1,\ldots,j-1}(v_j|u_1, \ldots, u_{j-1}) = \frac{1 + D_j \pm \sqrt{(1 + D_j)^2 - 4D_j v_j}}{2D_j}, \quad j \in \{2, \ldots, d\}, \tag{49}$$

where

$$D_j = \frac{\sum_{m=2}^{j} \sum_{1\leq n_1 \cdots \leq n_m = j} \theta_{n_1 \ldots n_m} (1-2u_{n_1}) \ldots (1-2u_{n_{m-1}})}{c_{j-1}(u_1, \ldots, u_{j-1})}, \quad j \in \{2, \ldots, d\}. \tag{50}$$

However, Ota and Kimura (2021) does not provide general indications for the sign of (49).

Proposition 9.1. *The valid solution of* (49) *is*

$$C^{-1}_{j|1,\ldots,j-1}(v_j|u_1, \ldots, u_{j-1}) = \frac{1 + D_j - \sqrt{(1+D_j)^2 - 4D_j v_j}}{2D_j}, \quad j \in \{2, \ldots, d\}.$$

Proof. The solution is valid if $C^{-1}_{j|1,\ldots,j-1}(v_j|u_1, \ldots, u_{j-1}) \in [0, 1]$. Consider $(u_1, \ldots, u_{j-1}) = (1/2, \ldots, 1/2)$. Then, the denominator of (50) is positive (since the combination is possible) and the numerator is equal to zero, hence $D_j = 0$. Evaluating (50) at $D_j = 0$ yields an indeterminate form, so we consider the limit of (49) as D_j approaches zero. We study the positive sign first. Select an index $1 \leq l \leq j-1$ such that $\theta_{lj} \neq 0$. Define the vector $\boldsymbol{u}_l = (u_1, \ldots, u_l, \ldots, u_j) = (1/2, \ldots, 1/2 - \text{sign}(\theta_{lj}) \times \varepsilon, \ldots, 1/2)$. One can compute the left limit $D_j \uparrow 0$ using the limit $\lim_{\epsilon \uparrow 0} D_j$, where D_j is evaluated at $\boldsymbol{u}_l$. One can also compute the right limit $D_j \downarrow 0$ using the limit $\lim_{\epsilon \downarrow 0} D_j$, where D_j is evaluated at $\boldsymbol{u}_l$. Applying l'Hôpital's rule, one has

$$\lim_{D_j \uparrow 0} \frac{1 + D_j + \sqrt{(1+D_j)^2 - 4D_j v_j}}{2D_j} = -\infty; \quad \lim_{D_j \downarrow 0} \frac{1 + D_j + \sqrt{(1+D_j)^2 - 4D_j v_j}}{2D_j} = \infty,$$

thus the limit does not exist and the positive sign is the incorrect solution. On the other hand (using l'Hôpital's rule, or multiplication by conjugate), we have

$$\lim_{D_j \to 0} \frac{1 + D_j - \sqrt{(1+D_j)^2 - 4D_j v_j}}{2D_j} = v_j \in [0, 1], \quad j \in \{1, \ldots, d\}.$$

□

With the revised proof, we present a first algorithm to generate random d-variate samples from a FGM copula defined by the parameters $\boldsymbol{\theta}$.

The algorithmic complexity of Algorithm 1 for a single sample is $O\left(2^d\right)$, which grows quickly for large d. For that reason, it is recommended to perform the simulation of random vectors in large dimensions using a stochastic model (see page 79 of Mai and Scherer (2014) for more detailed comments). This is now that the stochastic representation from Theorem 3.2 comes into play.

9.2 Stochastic model sampling method

The representation in (6) and the application of the integral probability transform allow us to put forward an efficient algorithm for sampling from the multivariate FGM copula. Let $\boldsymbol{Z}_0 = (Z_{1,0}, \ldots, Z_{d,0})$ and $\boldsymbol{Z}_1 = (Z_{1,1}, \ldots, Z_{d,1})$ be two independent random vectors of iid exponentially distributed rv where $Z_{j,0} \sim Exp(2)$ and $Z_{j,1} \sim Exp(1)$, for $j \in \{1, \ldots, d\}$. Then, Algorithm 2 samples $\boldsymbol{U}$ from a pmf $f_{\boldsymbol{I}}$.

Avoiding the exponential and the logarithmic operations, Algorithm 3 allows one to sample from $\boldsymbol{U}$ directly (applying Sklar's theorem earlier in the process). Let $\boldsymbol{V}_0 = (V_{1,0}, \ldots, V_{d,0})$ and $\boldsymbol{V}_1 = (V_{1,1}, \ldots, V_{d,1})$ be two independent random vectors of iid uniformly distributed rvs where $V_{j,0} \sim V_{j,0} \sim U(0, 1)$, for $j \in \{1, \ldots, d\}$.

Algorithm 1: Conditional sampling method for FGM

Input: Number of simulations n, parameters $\boldsymbol{\theta}$
Output: Set of simulations

for $l = 1, \ldots, n$ **do**
  Generate d independent $(0,1)$ uniformly distributed rvs $V_1^{(l)}, \ldots, V_d^{(l)}$;
  Set $U_1^{(l)} = V_1^{(l)}$;
  for $j = 2, \ldots, d$ **do**
    Compute $D_j^{(l)} = \frac{\sum_{m=2}^{j} \sum_{1 \leq n_1 \cdots \leq n_m = j} \theta_{n_1 \ldots n_m} (1-2U_{n_1}^{(l)}) \ldots (1-2U_{n_{m-1}}^{(l)})}{c_{j-1}(U_1^{(l)}, \ldots, U_{j-1}^{(l)})}.$;
    set $U_j^{(l)} = \frac{1 + D_j^{(l)} - \sqrt{\left(1 + D_j^{(l)}\right)^2 - 4 D_j^{(l)} V_j^{(l)}}}{2 D_j^{(l)}}$;
Return $\boldsymbol{U}^{(l)} = \left(U_1^{(l)}, \ldots, U_d^{(l)}\right)$.

Algorithm 2: Stochastic sampling method for FGM

Input: Number of simulations n, pmf $f_{\boldsymbol{I}}$
Output: Set of simulations

for $l = 1, \ldots, n$ **do**
  Generate n independent random vectors $\boldsymbol{I}^{(l)}$, $\boldsymbol{Z}_0^{(l)}$ and $\boldsymbol{Z}_1^{(l)}$;
  for $j = 1, \ldots, d$ **do**
    Compute $Y_j^{(l)} = Z_{j,0}^{(l)} + I_j^{(l)} Z_{j,1}^{(l)}$;
    Set $U_j^{(l)} = F_Y\left(Y_j^{(l)}\right) = 1 - e^{-Y_j^{(l)}}$;
Return $\boldsymbol{U}^{(l)} = \left(U_1^{(l)}, \ldots, U_d^{(l)}\right), l = 1, \ldots, n$.

Algorithm 3: Stochastic sampling method for FGM, alternative

Input: Number of simulations n, pmf $f_{\boldsymbol{I}}$
Output: Set of simulations

for $l = 1, \ldots, n$ **do**
  Generate the independent random vectors $\boldsymbol{I}^{(l)}$, $\boldsymbol{V}_0^{(l)}$ and $\boldsymbol{V}_1^{(l)}$;
  for $j = 1, \ldots, d$ **do**
    Set $U_j^{(l)} = 1 - \sqrt{1 - V_{j,0}^{(l)}} \times \left(1 - V_{j,1}^{(l)}\right)^{I_j^{(l)}}$;
Return $\boldsymbol{U}^{(l)} = \left(U_1^{(l)}, \ldots, U_d^{(l)}\right)$, $l = 1, \ldots, n$.

A great advantage of Algorithms 2 and 3 compared to Algorithm 1 lies on the simulation procedure of samples of the random vector $\boldsymbol{I}$. That procedure depends on the specification of the d-variate FGM copula. If one specifies the multivariate distribution of the random vector $\boldsymbol{I}$, one can sample observations using the stochastic representation of Theorem 3.2. If one specifies a FGM copula from the parameters $\boldsymbol{\theta}$, then one can either use Algorithm 1 or convert the natural representation into the stochastic representation by computing the pmf of $\boldsymbol{I}$ and then sampling from Algorithm 3. In addition, the conditional sampling method is sequential, meaning the expression of $C^{-1}_{j|1,\ldots,j-1}(v_j|u_1,\ldots,u_{j-1})$, for $j \in \{2,\ldots,d\}$ in (49) depends on the previous values $u_1,\ldots,u_{j-1}$; the latter does not lend itself to parallel computing. On the other hand, the stochastic representation depends on $\boldsymbol{I}$: after one computes $\boldsymbol{I}$, parallel computing is feasible.

9.3 Numerical illustration

In this section, we compare Algorithms 1 and 3 for subfamilies of FGM copulas. Some subfamilies admit alternate stochastic representations from the construction of $\boldsymbol{I}$ that make simulation even faster than Algorithm 3; in these cases we define new algorithms. Algorithm 1 is typically too long to compute for $d \geq 10$, so we omit computation times for these cases.

9.3.1 Unstructured vector of parameters

A first question of interest is as follows: when should one use Algorithm 1 over Algorithm 3? When given a vector of parameters $\boldsymbol{\theta}$, for which situation is the one-time computational cost of converting the parameters $\boldsymbol{\theta}$ into a pmf for $\boldsymbol{I}$ more efficient?

The first numerical example considers random parameters (and satisfying the constraints in (4)) for $d \in \{2,\ldots,6\}$. We use this example to study the sampling times and algorithmic complexities of Algorithms 1 and 3. When one defines the copula using the parameters $\boldsymbol{\theta}$, there is an added computational cost of converting the parameters to the pmf of the multivatiate Bernoulli distribution.

Algorithm 1 requires the computation of (49) and (50) for every sample. The total number of summation terms in (50) is $d^\star$, so the sample complexity for every simulation is $O(2^d)$. In total, the algorithmic complexity for M simulations is $O(M \times 2^d)$.

The algorithmic complexity of computing a single value of the pmf of $\boldsymbol{I}$ with (9) is $O(d^\star) = O(2^d)$, and there are 2^d values in the support of $\boldsymbol{I}$, so the total algorithmic complexity of converting a vector of $\boldsymbol{\theta}$ into the pmf of $\boldsymbol{I}$ is $O(4^d)$. However, this conversion is only required once, and is amortized (distributed) across samples when the number of simulations is high. Sampling from a vector of probabilities is $O(1)$ with a one-time setup of $O(2^d)$ using the alias method, see Walker (1977). Finally, sampling a single vector of rvs using Algorithm 3 has algorithmic complexity $O(d)$. We conclude that the total algorithmic complexity of sampling M observations using Algorithm 3 is $O(4^d) + O(2^d) + O(M \times d)$. Although Algorithm 3 requires a one-time computational burden of $O(4^d)$, we find that it is usually worth the effort for a large number of samples, since the initial setup is amortized.

Table 2 presents comparisons for the computation times of different sampling operations; all times are presented in seconds on a single core from a Intel® Core™ i5-7600K CPU @ 3.80GHz CPU using the R programming language R Core Team (2021). To use the stochastic representation within Algorithm 3, one must compute the pmf for the 2^d combinations of $\boldsymbol{i} \in \{0,1\}^d$. We present this computation time in the first row of Table 2. The second and third rows present the computation time to sample 10 000 observations from the set of parameters. The last row presents the extrapolated breakeven point, above which it is more efficient to perform the one-time conversion

	3	4	5	6
Conversion	0.01666	0.09056	0.45711	2.02756
Sampling (Algorithm 1)	0.05700	0.09200	0.29700	0.81600
Sampling (Algorithm 3)	0.01700	0.01000	0.01200	0.01200
Breakeven point	4 165	11 044	16 039	25 218

Table 2: Time comparisons (s) for the natural and stochastic representations of sampling operations

from the natural representation to the stochastic representation.

9.3.2 Extreme positive dependence

Sampling from the U^{EPD} is the fastest case for the stochastic representation since $\boldsymbol{I}^{+}$ admits positive pmfs for the two cases $\mathbf{0}_d$ and $\mathbf{1}_d$. Algorithm 4 is a simplification of Algorithm 3 to simulate values of $\boldsymbol{U}^{EPD}$.

Algorithm 4: Stochastic sampling method for U^{EDP}

Input: Number of simulations n

Output: Set of simulations

for $l = 1, \dots, n$ **do**

  Generate the independent random vectors $\boldsymbol{V}_0^{(l)}$ and $\boldsymbol{V}_1^{(l)}$;

  Generate $I^{(l)} \sim Bern\left(\frac{1}{2}\right)$;

  for $j = 1, \dots, d$ **do**

    Set $U_j^{(l)} = 1 - \sqrt{1 - V_{j,0}^{(l)}} \times \left(1 - V_{j,1}^{(l)}\right)^{I^{(l)}}$, for $j \in \{1, \dots, d\}$.;

Return $\boldsymbol{U}^{(\boldsymbol{l})} = \left(U_1^{(l)}, \dots, U_d^{(l)}\right), l = 1, \dots, n$.

Sampling from the EPD copula with Algorithm 1 admits a faster implementation (although not as significant as in the stochastic representation); one can skip computations since the k-dependence parameters with $k \in \{3, 5, 7, \dots\}$ are zero. Table 3 compares the simulation times, in seconds. One notices a large advantage of Algorithm 4 in terms of computation time.

9.3.3 Markov-Bernoulli model

For the subfamily where the random vector $\boldsymbol{I}$ is defined from the Markov chain with binary state space (Markov-Bernoulli model of Section 8) with pmf as in (45), one can sample $\boldsymbol{I}$ iteratively using the transition probability matrix starting with $I_1 \sim Bern(1/2)$, as shown in Algorithm 5. In Table 4, we present the simulation times for the different algorithms.

d	3	4	5	6	7	10	15	20
Algorithm 1	0.00154	0.00521	0.01485	0.03789	0.08954	–	–	–
Algorithm 4	0.00039	0.00052	0.00046	0.00052	0.00057	0.00081	0.00115	0.00146

Table 3: Computation times (s) for 1000 simulations of C_d^{EPD}

Algorithm 5: Stochastic sampling method for the Markov-Bernoulli FGM copula

Input: Number of simulations n, transition matrix $\boldsymbol{P}$

Output: Set of simulations

Generate the independent random vectors $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$;

for $l = 1, \ldots, n$ **do**

  Generate $I_1^{(l)} \sim Bern(\frac{1}{2})$;

  for $j = 2, \ldots, d$ **do**

    Generate $I_j^{(l)} \sim Bern\left(p_{I_{j-1}^{(l)}|1}\right)$;

  for $j = 1, \ldots, d$ **do**

    Set $U_j^{(l)} = 1 - \sqrt{1 - V_{j,0}^{(l)}} \times \left(1 - V_{j,1}^{(l)}\right)^{I_j^{(l)}}$;

Return $\boldsymbol{U}^{(l)} = \left(U_1^{(l)}, \ldots, U_d^{(l)}\right), l = 1, \ldots, n.$

d	3	4	5	6	7	10	15	20
Algorithm 1	0.00278	0.01101	0.03089	0.08111	0.19773	NA	NA	NA
Algorithm 3	0.00194	0.00206	0.00231	0.00267	0.00274	0.00404	0.01159	0.12132
Algorithm 5	0.00060	0.00060	0.00073	0.00092	0.00098	0.00140	0.00206	0.00244

Table 4: Computation times (s) for 1000 simulations for Markov-Bernoulli and $\alpha = 0.5$.

9.4 Conclusions on the numerical study

We offer the following observations from the numerical study of sampling with FGM copulas. Starting from a vector of parameters $\boldsymbol{\theta}$, the one-time conversion into the pmf of $\boldsymbol{I}$ is the only disadvantage of using Algorithm 3. There is a breakeven point for converting the parameters, but for high enough samples, this conversion is worth it.

The conversion is only necessary when one defines a FGM copula using the natural representation. As stated in Remarks 3.5 and 3.6, the representation using $\boldsymbol{\theta}$ is undesirable for large d, while one can still obtain tractable and interpretable FGM copulas with high d using a single parameter with Markov-Bernoulli model (or other symmetric multivariate Bernoulli random vectors). As d increases, it is easier to define a FGM copula using the stochastic representation with existing symmetric multivariate Bernoulli rvs than with the natural representation.

The subfamilies where $\boldsymbol{I}$ can be sampled from a stochastic representation yield the fastest sampling times. One has considerable advantages of selecting a family of symmetric multivariate Bernoulli distributions that is simple to sample when dealing with high-dimensional FGM copulas.

10 Proofs of Section 2

We first offer the following Lemma that demonstrates the univariate representation.

Lemma 10.1. *Let the random variable (rv) I follow a symmetric Bernoulli distribution. Let Z_0 and Z_1 be two exponentially distributed rvs with cdf $F_{Z_l} = 1 - e^{-(2-l)x}$, $x \geq 0$, for $l \in \{0, 1\}$. Let the rv Y be defined as $Y = Z_0 + I \times Z_1$ where I, Z_0, and Z_1 are independent. Then, $Y \sim Exp(1)$.*

Proof of Lemma 10.1. We have

$$F_{Y|I=0}(x) = F_{Z_0}(x) = 1 - e^{-2x} = (1 - e^{-x})(1 + e^{-x}), \quad x \geq 0, \tag{51}$$

and

$$F_{Y|I=1}(x) = F_{Z_0+Z_1}(x) = (1-e^{-x})^2 = (1-e^{-x})(1-e^{-x}), \quad x \geq 0. \tag{52}$$

Combining (51) and (52), we obtain

$$F_{Y|I=i}(x) = F_{Z_0+i\times Z_1}(x) = (1-e^{-x})(1+(-1)^i e^{-x}), \quad x \geq 0, \quad i \in \{0,1\}. \tag{53}$$

Conditioning on the rv I, using (53), and given that $E[(-1)^I] = 0$, we find

$$F_Y(x) = E[F_{Y|I}(x)] = (1-e^{-x})(1+e^{-x}E[(-1)^I]) = 1-e^{-x}, \quad x \geq 0. \tag{54}$$

□

Proof of Theorem 3.1. We use Lemma 10.1 to identify the marginal distributions of the rvs $Y_j, j \in \{1, \ldots, d\}$. Conditioning on the possible values of $\boldsymbol{I}$ and since the random vectors $\boldsymbol{I}$, $\boldsymbol{Z}_0$, and $\boldsymbol{Z}_1$ are assumed independent, we first obtain the following joint cdf of $\boldsymbol{Y}$:

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{j=1}^{d} F_{Z_{j,0}+i_j Z_{j,0}}(x_j), \quad \boldsymbol{x} \in \mathbb{R}_+^d. \tag{55}$$

Inserting (53) in (55) leads to the desired result. □

We prove the main theorem using multivariate exponential FGM distributions, but one could also prove it using the d-variate FGM copula directly. By Theorem 2.2, both approaches are valid. The proof has two steps. First, we show that the cdfs using the stochastic representation is equivalent to the cdf using the natural representation. Then, we show that both classes of valid parameters are equivalent.

Proof of Theorem 3.2. Rewrite (7) in Theorem 3.1 as

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \prod_{m=1}^{d}\left(1-e^{-x_m}\right)\left(\sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{j=1}^{d}\left(1+(-1)^{i_j}e^{-x_j}\right)\right),$$

which becomes

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \left(\prod_{m=1}^{d}\left(1-e^{-x_m}\right)\right) E_{\boldsymbol{I}}\left[\prod_{j=1}^{d}\left(1+(-1)^{I_j}e^{-x_j}\right)\right].$$

Expanding the product yields

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \left(\prod_{m=1}^{d}\left(1-e^{-x_m}\right)\right) E_{\boldsymbol{I}}\left[1+\sum_{k=1}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d}(-1)^{I_{j_1}+\cdots+I_{j_k}}e^{-x_{j_1}}\ldots e^{-x_{j_k}}\right]. \tag{56}$$

By symmetry of the distribution of I, we have $E\left[(-1)^I\right] = 0$ and

$$F_{\boldsymbol{Y}}(\boldsymbol{x}) = \left(\prod_{m=1}^{d}\left(1-e^{-x_m}\right)\right)\left(1+\sum_{k=2}^{d}\sum_{1\leq j_1<\cdots<j_k\leq d} E_{\boldsymbol{I}}\left[(-1)^{I_{j_1}+\cdots+I_{j_k}}\right]e^{-x_{j_1}}\ldots e^{-x_{j_k}}\right). \tag{57}$$

Next, one recognizes that $(-1)^I$ is also $1-2I$ and obtains

$$E_{\boldsymbol{I}}\left[(-1)^{I_{j_1}+\cdots+I_{j_k}}\right] = E_{\boldsymbol{I}}\left[\prod_{l=1}^{k}(-1)^{I_{j_l}}\right] = E_{\boldsymbol{I}}\left[\prod_{l=1}^{k}(1-2I_{j_l})\right] = (-2)^k E_{\boldsymbol{I}}\left[\prod_{l=1}^{k}\left(I_{j_l}-\frac{1}{2}\right)\right]. \quad (58)$$

Writing $\theta_{j_1\cdots j_k} = (-2)^k E_{\boldsymbol{I}}\left[\prod_{l=1}^{k}\left(I_{j_l}-\frac{1}{2}\right)\right]$ and inserting in (57) completes the bijection.

It remains to prove that both admissible sets of parameters coincide, that is, for every $\boldsymbol{\theta} \in \mathcal{T}_d$ and applying (9) yields a valid pmf $f_{\boldsymbol{I}} \in \mathcal{B}_d$, and for every $f_{\boldsymbol{I}} \in \mathcal{B}_d$ and applying (10) yields a valid set of parameters $\boldsymbol{\theta} \in \mathcal{T}_d$.

From Theorem 2 of Sharakhmetov and Ibragimov (2002), one can express joint distributions of two-valued random variables as

$$p(x_1,\ldots,x_d) = \prod_{k=1}^{d} p_k(x_k)\left(1+\sum_{k=2}^{d}\sum_{1\le j_1<\cdots<j_k\le d}\alpha_{j_1\ldots j_k}\prod_{l=1}^{k}(x_{j_l}-a_{j_l}p_{j_k}-b_{j_l}q_{j_l})\right),$$

$x_k \in \{a_k, b_k\}, k \in \{1,\ldots,d\}$, with $p_k = p_k(a_k)$ and $q_k = p_k(b_k) = 1-p_k(a_k)$. From Theorem 1 of Sharakhmetov and Ibragimov (2002), we have

$$\alpha_{j_1\ldots j_k} = E\left[\prod_{l=1}^{k}\frac{X_{j_l}-E[X_{j_l}]}{Var(X_{j_l})}\right],$$

for $1 \le j_1 < \cdots < j_k \le d$ and $k \in \{2,\ldots,d\}$. Also, the parameters must satisfy

$$\sum_{k=2}^{d}\sum_{1\le j_1<\ldots j_k\le d}\alpha_{j_1\ldots j_k}\prod_{l=1}^{k}(x_{j_l}-a_{j_l}p_{j_k}-b_{j_l}q_{j_l}) \ge -1, \quad (59)$$

for $x_k \in \{a_k, b_k\}$. Substituting $a_k = 0, b_k = 1, p_k = 1/2, k \in \{1,\ldots,d\}$, the constraints become

$$\sum_{k=2}^{d}\sum_{1\le j_1<\ldots j_k\le d}\alpha_{j_1\ldots j_k}\prod_{l=1}^{k}\left(x_{j_l}-\frac{1}{2}\right) \ge -1 \quad (60)$$

$$\sum_{k=2}^{d}\sum_{1\le j_1<\ldots j_k\le d}4^k E\left[\prod_{l=1}^{k}(X_{j_l}-E[X_{j_l}])\right]\left(-\frac{1}{2}\right)^k\prod_{l=1}^{k}(2x_{j_l}-1) \ge -1$$

$$1+\sum_{k=2}^{d}\sum_{1\le j_1<\ldots j_k\le d}(-2)^k E\left[\prod_{l=1}^{k}(X_{j_l}-E[X_{j_l}])\right]\prod_{l=1}^{k}(2x_{j_l}-1) \ge 0,$$

for $x_k \in \{0,1\}$ and $k \in \{1,\ldots,d\}$. Inserting (8) and observing that $(2x_k-1), x_k \in \{0,1\}$ and $k \in \{1,\ldots,d\}$ is equivalent to $\varepsilon_k \in \{-1,1\}$, both (60) and (4) span the same constraints. See also Fontana and Semeraro (2018) for an alternate proof that both admissible sets of parameters correspond. □

11 Proof of Proposition 8.1

The expression for the joint pmf of the vector of rvs $\boldsymbol{I}$ is

$$\begin{aligned} f_{\boldsymbol{I}}(\boldsymbol{i}) &= \Pr(I_1 = i_1) \prod_{m=2}^{d} p_{i_m|i_{m-1}} \\ &= \frac{1}{2} \prod_{m=2}^{d} \left(\frac{1+\alpha}{2} \times 1_{\{i_m = i_{m-1}\}} + \frac{1-\alpha}{2} \times 1_{\{i_m \neq i_{m-1}\}} \right) \\ &= \frac{1}{2^d} \prod_{m=2}^{d} \left(1 + (-1)^{|i_m - i_{m-1}|} \alpha \right), \end{aligned} \tag{61}$$

which becomes

$$f_{\boldsymbol{I}}(\boldsymbol{i}) = \frac{1}{2^d} \left(1 + \sum_{l=2}^{d} \sum_{1 \leq j_1 < \cdots < j_l \leq d} (-1)^{|i_{j_2} - i_{j_1}| + \cdots + |i_{j_l} - i_{j_{l-1}}|} \alpha^l \right), \tag{62}$$

for $\boldsymbol{i} \in \{0,1\}^d$. Also, for $1 \leq j_1 < \ldots < j_k \leq d$ and $k \in \{2, \ldots, d\}$, the expression for the joint pmf of the vector of rvs $(I_{j_1}, \ldots, I_{j_k})$ is given by

$$\begin{aligned} f_{I_{j_1},\ldots,I_{j_k}}(i_{j_1}, \ldots, i_{j_k}) &= \Pr(I_{j_1} = i_{j_1}) \prod_{m=2}^{k} p_{i_{j_m}|i_{j_{m-1}}} \\ &= \frac{1}{2} \prod_{m=2}^{k} \left(\frac{1+\alpha^{j_m - j_{m-1}}}{2} \times 1_{\{i_{j_m} = i_{j_{m-1}}\}} + \frac{1-\alpha^{j_m - j_{m-1}}}{2} \times 1_{\{i_{j_m} \neq i_{j_{m-1}}\}} \right) \\ &= \frac{1}{2^k} \prod_{m=2}^{k} \left(1 + (-1)^{|i_{j_m} - i_{j_{m-1}}|} \alpha^{j_m - j_{m-1}} \right), \end{aligned} \tag{63}$$

which becomes

$$f_{I_{j_1},\ldots,I_{j_k}}(i_{j_1}, \ldots, i_{j_k}) = \frac{1}{2^k} \left(1 + \sum_{l=2}^{k} \sum_{1 \leq n_1 < \cdots < n_l \leq k} (-1)^{|i_{j_{n_2}} - i_{j_{n_1}}| + \cdots + |i_{j_{n_l}} - i_{j_{n_{l-1}}}|} \alpha^{j_{n_l} - j_{n_1}} \right), \tag{64}$$

for $(i_{j_1}, \ldots, i_{j_k}) \in \{0,1\}^k$. Combining (5), (10) and (64) yields the first result. The result of (47) follows by replacing (61) within (11).

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